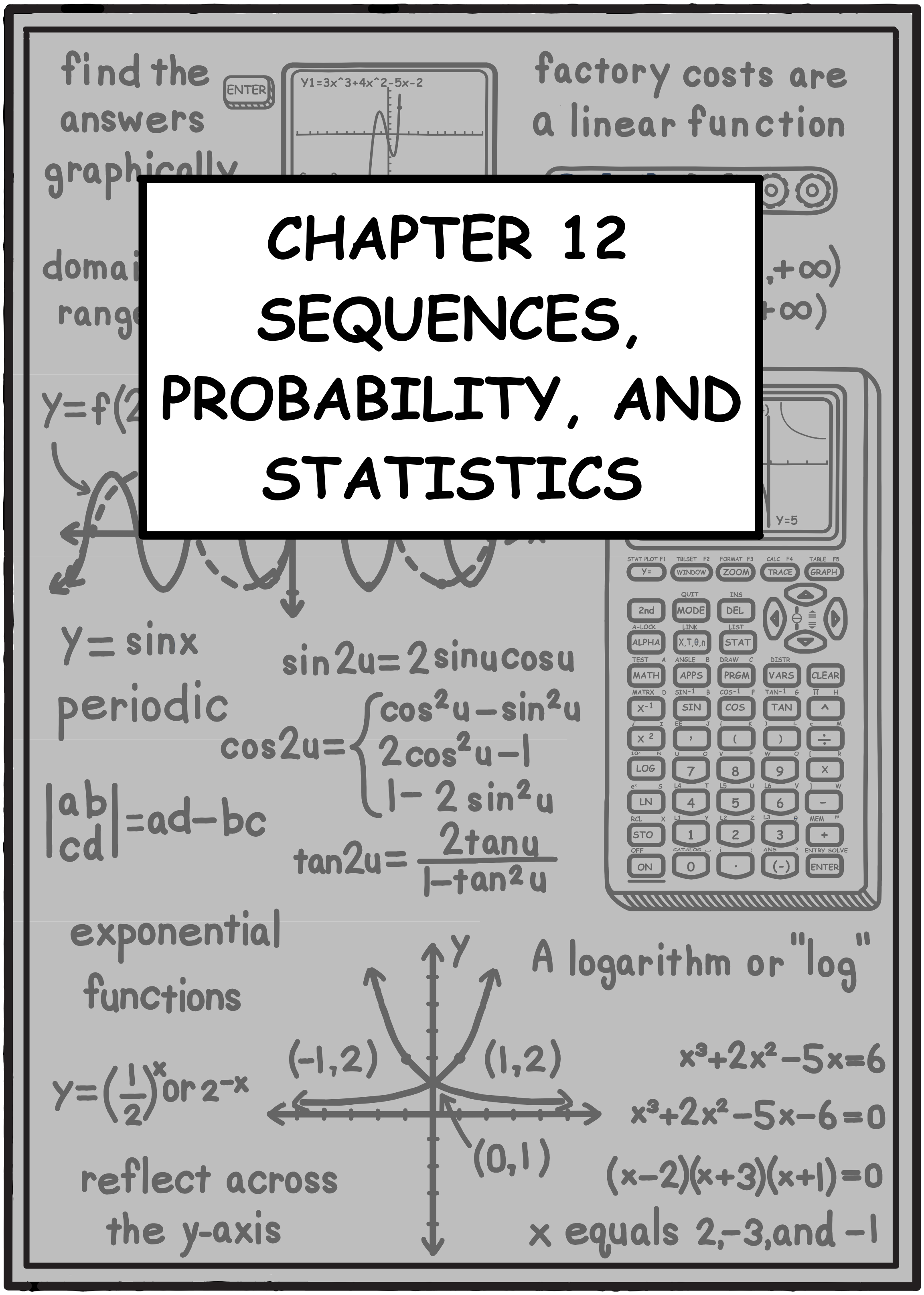
CHAPTER 12 SEQUENCES, PROBABILITY, AND STATISTICS



Lesson 77—Sequences

We've finished with conic sections, so now we're going to switch gears and talk about sequences. In simple language, a sequence is just a list of numbers that progress in a certain pattern. Here are two examples.

$$2, 4, 6, 8, 10, 12$$

$$3, 9, 27, 81, 243, \ldots\ 3^n$$

On top, the numbers start at 2 and then increase by 2 each time. On bottom, the numbers are 3^1, 3^2, 3^3 and on up. So the exponent on the 3 increases by 1 each time. But both sequences show a pattern. One big difference between the two, though, is that the top sequence is finite; it stops with the number 12. However, the bottom sequence is infinite, since the numbers just keep going. That's what the dots mean. The last term, 3^n, just shows the formula that's used to generate all of the numbers in the sequence. The first term 3, is 3^1. Next comes 9, which is 3^2, and so on. The n can represent any positive integer. The numbers of a sequence are called terms, by the way. And the term 3^n is sometimes called the general term. Here are two more examples of sequences, both of which are infinite.

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$$

$$0, 1, 3, 6, 10, 15, 21, 28, 36, 45, \ldots$$

Sequences as Functions

A sequence can qualify as a function. Remember, the technical definition of a function is that it can have just one output (y-value) for each input (x-value). We can think of the terms of the sequence as the y-values. They're the range. Then the x-values are just the order in which the terms are listed. For example, here are the x- and y-values for the sequence 3^n.

Table 77.1

y-values	3,9,27,81,243,...3^n
x-values	1,2, 3, 4, 5,... n

Since no x-value has two matching y-values, this sequence qualifies as a function.

The big difference between a sequence that's a function and most of the functions we've been working with is that a sequence only allows positive integers in its domain. That's why we use the letter n to stand for the independent variable. It's generally understood that the letter n includes just positive integers. (The letter k is also sometimes used.) Restricting the domain to positive integers limits the range as well.

The symbols for a sequence are a little different from those for a function. Instead of writing $f(n) = 3^n$, we usually represent each term of the sequence by a letter with a little n below to designate the order in which the term appears. For example, the first term of 3^n is a_1 and the second term is a_2. So we write $a_1 = 3^1 = 3$ and $a_2 = 3^2 = 9$. The general symbol for a term of the sequence is just a_n. Other letters besides a can be used as well.

Sequences and Formulas

It's possible to use a formula to represent an entire sequence. Instead of writing 3, 9, 27, 81, 243, … 3^n, for instance, we could just write 3^n. That tells us the pattern of the sequence, which is everything we need to know. We can use 3^n to find any specific term of the sequence we want. To find the 5th term, we just put 5 in for *n* like this.

$$a_5 = 3^5 = 243$$

We can find any other term the same way. This is called finding the "*n* th term" of a sequence.

Another example of a formula is $4n-1$. This represents the sequence below.

$4n-1$ values	3,	7,	11	15,	19, …
n-values	1,	2,	3,	4,	5, ...

To find the 47th term of $4n-1$, we just put 47 in for *n* and calculate the value of the formula.

$$a_{47} = 4(47) - 1 = 187$$

Think of how much longer it would have taken to find this answer by just listing the terms of the sequence one-by-one, according to the pattern.

Most sequences have patterns where the next term depends on the value of the previous term. For instance, in the sequence 3, 7, 11, 15, 19 …, the next term can be calculated by just adding 4 to the previous term. The direct formula for this sequence is $4n-1$, as we showed. But we can also write a formula for the sequence using the previous term as the independent variable. If a_n represents any particular term of the series, then $a_n = a_{n-1} + 4$. This formula can also be used to generate specific terms of the sequence. Starting with the term 19, we can find the very next term like this.

$$a_n = a_{n-1} + 4$$

$$23 = 19 + 4$$

A formula that uses a previous term for its independent variable is called a recursive formula. Recursive formulas are usually a little easier to write (when given several terms of the sequence), but they're not as useful as direct formulas. With a recursive formula, for instance, we can't find the 47th term of a sequence just by plugging in 47 for the independent variable.

Arithmetic Sequences

There are a couple of types of sequences that are really important because they're used so much in the real world. First, there's an **arithmetic sequence**, which is just a sequence where each term is obtained by adding the same number to the previous term. The sequences 2, 4, 6, 8, 10, 12 and 3, 7, 11, 15, 19 …, are both arithmetic. The key feature of an arithmetic sequence is that the difference between any two terms is always the same. This is called the **common difference**. So the common difference of the sequence 2, 4, 6, 8, 10, 12 is 2 and the common difference of the sequence 3, 7, 11, 15, 19 … is 4.

2, 4, 6, 8, 10, 12	**common difference of 2**
3, 7, 11, 15, 19 …	**common difference of 4**

More generally, any arithmetic sequence can be written in the form below, where a is the first term and d is the common difference.

$$\{a, a+d, a+2d, a+3d, ..., a+(n-1)d...\}$$ **common difference of d**

That means in general the direct formula for any arithmetic sequence is $a_n = a_1 + (n-1)d$. So if we're trying to find the 12th term of some arithmetic sequence that starts with the number 9 and has a common difference of 4, we just put 9 in for a_1 and 4 in for d to get this.

$$a_n = 9 + (12-1)(4) = 53$$

The general form of the recursive formula for an arithmetic sequence is written like this.

$$a_n = a_{n-1} + d \quad (\text{where } n \geq 2)$$

If a_n is some term of the sequence, then a_{n-1} is the previous term. Adding the common difference (d) to the previous term gives us a_n. That's all the recursive formula is saying.

The easiest way to tell whether a sequence is arithmetic is just to subtract several pairs of consecutive terms and see if you get the same answer each time. If you do, then the sequence is arithmetic. If the differences are not all the same, then the sequence isn't arithmetic.

Geometric Sequences

The second important type of sequence is a **geometric sequence**. This is where the terms increase by the same factor each time. The sequence 3^n is geometric, because to get each term we can just multiply the previous term by 3.

$$\begin{array}{ccccc} 3, & 9, & 27, & 81, & 243, \ldots \\ 3, & 3\cdot 3, & 9\cdot 3, & 27\cdot 3, & 81\cdot 3, \ldots \end{array}$$

Here's another example. In this one each term is 2 times the value of the previous term.

$$\begin{array}{ccccc} 2, & 4, & 8, & 16, & 32, \ldots \\ 2, & 2\cdot 2, & 4\cdot 2, & 8\cdot 2, & 16\cdot 2, \ldots \end{array}$$

The direct formula for this sequence is 2^n, and the recursive formula is $a_n = 2a_{n-1}$ (where $n \geq 2$).

So in a geometric sequence, instead of adding each previous term by the same number, we multiply each previous term by the same number. The number we need to multiply by is called the **common ratio**. The common ratio of 2, 4, 8, 16, 32, 64, … is 2. The name comes from the fact that the ratio of any two consecutive terms in the sequence is always equal to the common ratio. For instance, dividing any term in 2, 4, 8, 16, 32, 64, … by the previous term always gives an answer of 2.

$$\frac{4}{2} = 2, \ \frac{8}{4} = 2, \ \frac{16}{8} = 2, \ \frac{32}{16} = 2, \ \frac{64}{32} = 2$$ **common ratio = 2**

This means the easy way to recognize a geometric sequence is just to divide several pairs of consecutive terms and see if you get the same number every time. There are lots of other geometric sequences with different starting points and different constant ratios. Here are a few more.

$$3, 6, 12, 24, 48, 96, \ldots 3\cdot 2^{n-1}$$

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots \left(\frac{1}{2}\right)^n$$

$$1, -3, 9, -27, 81, -243, \ldots (-3)^{n-1}$$

Notice that the top sequence has a constant 3 in front of the formula. That's necessary in order to make the first term of the sequence equal to 3. After that each term is twice the size of the previous one. If the first term of the sequence had been 4, then the number in front of 2^{n-1} would have been 4. The exponent is $n-1$ so that the first exponent will equal 0. See, putting 1 in for n gives this: $3\cdot 2^{n-1} = 3\cdot 2^{1-1} = 3\cdot 2^0 = 3$. The domain of a sequence is always the positive integers, so you sometimes have to make adjustments like this to start the sequence of with the proper first term.

The sequence, $1, -3, 9, -27, 81, -243, \ldots (-3)^{n-1}$ is also interesting. It has a negative common ratio: -3. This causes the sign of the terms to alternate between positive and negative. And notice that the $n-1$ in the exponent of the formula $(-3)^{n-1}$ causes the first term to equal 1, since $(-3)^0 = 1$. If the exponent were just n, then the first term of the sequence would equal -3, because $(-3)^1 = -3$.

We can use the formula of a geometric sequence to find n^{th} term, just as we do with other sequences. For instance, to find the 7th term of $\left(\frac{1}{2}\right)^n$, we just put 7 in for n and calculate the value of the expression: $a_7 = \left(\frac{1}{2}\right)^7 = \frac{1}{128}$.

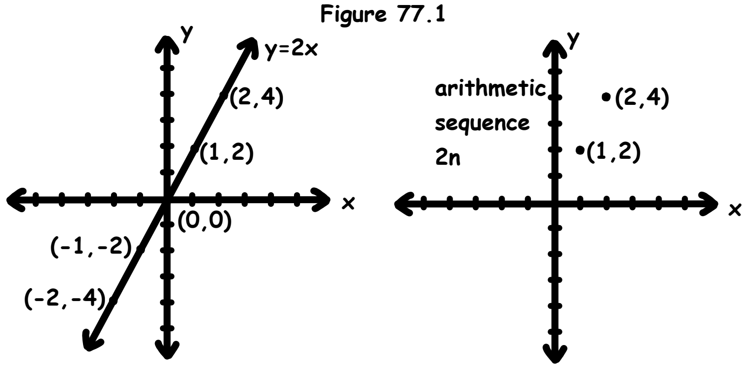
Geometric sequences can also be shown with general symbols. The general for any geometric sequence is written like this (with a common ratio of r).

$$\{a, ar, ar^2, ar^3, \ldots, ar^{n-1} \ldots\}$$ **common ratio = r**

The general form of a recursive formula for any geometric sequence is $a_n = ra_{n-1}$, (where $n \geq 2$).

Arithmetic sequences have the same form as linear functions. An easy way to see this is to compare the graphs of the two. Here are the graphs of the arithmetic sequence $2n$ and the linear function $y = 2x$.

Figure 77.1

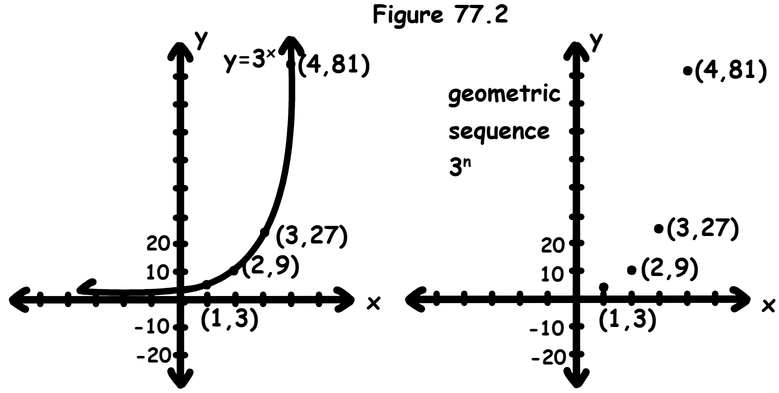


Only integers are in the domain

The only difference in the two functions is their domains. The sequence domain is limited to positive integers, as always. That causes its graph to include only a series of points. However, the function's domain includes all real numbers, which makes its graph a smooth, continuous line.

The same kind of relationship exists between a geometric sequence and an exponential function. For instance, compare the graphs of the geometric sequence 3^n and $y = 3^x$.

Figure 77.2



Only integers are in the domain

See, the sequence just has points, but the function is a smooth curve. That's because the sequence function has a domain that's limited to positive integers.

Practice 77

a. Find the next three terms in the sequence : $9,\ -3,\ 1,\ -\frac{1}{3}\ ...$

b. Write a direct formula (n^{th} term) for the geometric sequence 1, 5, 25, 125, 625, …?

A. $2n+3$ B. 5^n C. $n+4$
D. $5n$ E. 5^{n-1}

c. Find the 7^{th} term in the geometric sequence with $a_1 = 3$ and a common ratio (r) of 4.

d. Which of the following equations has roots of 4 and $-\frac{1}{2}$?

A. $2x^2 - 9x + 4 = 0$ B. $2x^3 + 3x^2 - 18x + 8 = 0$ C. $2x^2 - 7x - 4 = 0$
D. $2x^2 + 7x - 4 = 0$ E. $2x^2 + 9x + 4 = 0$

e. If $8\sin^2\theta + 2\sin\theta - 1 = 0$, then what is the smallest positive value of θ in degrees? Estimate the answer to one decimal place.

f. Solve the word problem below.

The cost of a booth at a county fair can be represented by the function $C(t) = 1{,}000 - 25t$ where t is the number of years that the company has rented a booth. Based on this model, what would be the price of a booth for a company who has rented a booth for the past 6 years?

Problem Set 77

Tell whether each sentence below is True or False.

1. An arithmetic sequence is a sequence where each term is obtained by adding the same number to the previous term.

2. A geometric sequence is a sequence where the terms are increased by the same factor (multiplied by the same number) each time.

Answer each question below.

3. Select the center and vertices of $\frac{y^2}{15^2} - \frac{x^2}{8^2} = 1$

A. Center $(15,0)$; Vertices $(-8,0)$, $(8,0)$
B. Center $(0,0)$; Vertices $(0,-8)$, $(0,8)$
C. Center $(0,0)$; Vertices $(0,-15)$, $(0,15)$
D. Center $(0,8)$; Vertices $(0,15)$, $(15,0)$
E. Center $(0,0)$; Vertices $(-15,0)$, $(15,0)$

4. Select the center and vertices of $\frac{(x+6)^2}{49} - \frac{(y-4)^2}{9} = 1$

A. Center $(6,-4)$; Vertices $(-3,0)$, $(3,0)$
B. Center $(-6,4)$; Vertices $(-1,-4)$, $(13,-4)$
C. Center $(0,0)$; Vertices $(7,0)$, $(0,3)$
D. Center $(-6,4)$; Vertices $(-6,1)$, $(-6,7)$
E. Center $(-6,4)$; Vertices $(-13,4)$, $(1,4)$

5. If the x-axis is translated 5 places down and the y-axis 4 places to the right, the equation $y = -\frac{1}{2}x - 3$ changes to which of the following?

A. $y' = -\frac{1}{2}x' - 6$
B. $y' = -\frac{1}{2}x'$
C. $y' = x'$
D. $y' = -2x'$
E. $y' = -\frac{1}{2}x' - \frac{3}{2}$

Tell whether each conic section below is a circle, parabola, ellipse or hyperbola without completing the square?

6. $6x^2 + 12y^2 - 84x + 72y - 53 = 0$

7. $10x^2 - 60x - 8y^2 + 144y + 135 = 0$

Find the next three terms in each sequence below.

8. 4, 8, 12, 16 …

9. 3, 15, 75, 375 …

(a) 10. 4, -2, 1, $-\frac{1}{2}$ ...

Select the correct answer for each sequence below.

11. Write a direct formula (n^{th} term) for the arithmetic sequence 7, 14, 21, 28, …?

A. $7n-7$ B. 7^n C. 7^{n-1}
D. $n+7$ E. $7n$

(b) 12. Write a direct formula (n^{th} term) for the geometric sequence 1, 4 , 16, 64 , 256, …?

A. $4n$ B. 4^n C. $n+4$
D. 4^{n-1} E. $n+3$

13. Write a recursive formula for the arithmetic sequence 2, 5, 8, 11, …?

A. $a_n = 2n$ B. $a_n = a_{n-1} + 3$ C. $a_n = 3a_{n-1}$
D. $a_n = 2a_{n-1} + 1$ E. $a_n = 2^n$

14. Write a recursive formula for the arithmetic sequence 3, 1, $\frac{1}{3}$, $\frac{1}{9}$, …?

A. $a_n = \frac{1}{3}a_{n-1}$ B. $a_n = \left(\frac{1}{3}\right)^n$ C. $a_n = \frac{1}{3}n$
D. $a_n = 2a_{n-1} - 5$ E. $a_n = a_{n-1} - 2$

Answer each question below.

15. Find the 17^{th} term in the arithmetic sequence with $a_1 = 5$ and a common difference (d) of -2.

(c) 16. Find the 6^{th} term in the geometric sequence with $a_1 = 2$ and a common ratio (r) of 3.

17. Find the common ratio (r) in the geometric sequence: $1, -\frac{1}{4}, \frac{1}{16}, \ldots$

Select the answer for each question below.

18. If $f(x) = \sqrt{x}$ and $f(g(x)) = 3\sqrt{x}$, then $g(x) = ?$

A. $3x^2$ B. $3x$ C. $9x$
D. x^3 E. $\frac{x}{3}$

(d) 19. Which of the following equations has roots of 3 and $-\frac{1}{2}$?

A. $2x^2 + 5x - 3 = 0$ B. $2x^2 + 7x + 3 = 0$ C. $2x^3 + 5x^2 - 4x - 3 = 0$
D. $2x^2 - 7x + 3 = 0$ E. $2x^2 - 5x - 3 = 0$

20. If $f(k) = 25^{-k}$, then $f\left(-\frac{1}{4}\right) = ?$

A. 25
B. 5
C. $25^{-\frac{1}{4}}$
D. $\sqrt{5}$
E. $\frac{1}{25}$

Answer the question below. Estimate the answer to one decimal place.

(e) 21. If $6\sin^2\alpha + \sin\alpha - 1 = 0$, then what is the smallest positive value of α in degrees?

Solve the word problem below.

(f) 22. The annual cost of belonging to a certain country club can be represented by the function $C(t) = 7{,}000 - 100t$ where *t* is the number of years a member has belonged to the country club. Based on this model, how much would the annual cost be for a member who had belonged for 15 years?

Lesson 78—The Sum of a Sequence

In the last lesson, we started learning about sequences, which are just lists of numbers that progress in a pattern. Sometimes it's useful to be able to calculate the sum of the terms in a sequence. Here's a simple example.

$$2, 4, 6, 8, 10, 12, 14, \ldots, 2n$$

Let's find the sum of the first seven terms of this sequence. All we have to do is add those terms together to get 56.

$$2+4+6+8+10+12+14=56$$

Sigma Notation

There's a short way to write the sum of a sequence. We use the capital Greek letter sigma, which is written $\sum$. This is the Greek capital "*S*," so sigma stands for "sum." To show the sum of the sequence above, we write $n=1$ below $\sum$. That says we're starting with the first term of the sequence. Then we write 7 above $\sum$, to say that we're ending with the 7th term of the sequence.

$$\sum_{n=1}^{7} 2n = 56$$

The expression $\sum_{n=1}^{7} 2n$ means "Start with the first term of the sequence $2n$ and add all the terms up through the 7th term." And we already saw that that sum equals 56. Here are several other examples of showing sums of sequences using sigma notation.

$$\sum_{n=1}^{8} n^2 = 1^2+2^2+3^2+4^2+5^2+6^2+7^2+8^2 = 204$$

$$\sum_{n=1}^{4} 3^n = 3^1+3^2+3^3+3^4 = 120$$

$$\sum_{k=1}^{5} \frac{1}{k} = \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5} = \frac{137}{60}$$

Formulas for Sums

Since arithmetic sequences are so common, mathematicians have worked out formulas for calculating their sums. If $\{a_1, a_2, a_3, \ldots, a_n\}$ is some finite arithmetic sequence, then the sum of its terms can be calculated with the formula below.

$$\frac{n}{2}(a_1+a_n) \qquad \textbf{sum of arithmetic sequence}$$

This looks kind of complicated, but the expression $\frac{n}{2}(a_1+a_n)$ is just saying to add the first and last term of the sequence whose sum is being calculated. Then multiply that total by the number of terms divided by 2. To see if it works, let's try the formula on a sequence that starts with 5 and has a common difference is 2. The first 6 terms of this sequence have a sum of 60.

$$5, 7, 9, 11, 13, 15$$

$$5+7+9+11+13+15=60$$

Let's see if we get the right answer with the formula: $\frac{n}{2}(a_1+a_n)$ equals $\frac{6}{2}(5+15)$, which comes out to 60. It works.

There's a well-known story about this formula.[1] When Friedrich Gauss, one of the greatest mathematical geniuses of all time, was just 10 years old, his teacher assigned everyone in his school class the task of adding all of the whole numbers from 1 to 100 (just to keep everyone busy). After just a few seconds, Gauss wrote down the answer and handed it to the teacher. Assuming that Gauss had just guessed, the teacher didn't even bother looking at the answer, and made the boy sit back down and continue working. Later that day, the teacher happened to glance at the answer and was astonished to find the number 5,050, which is exactly right. Gauss had realized that the numbers 1 through 100 can be paired up so that every pair has a sum of 101. All you have to do is take the first number plus the last number ($1+100$) and work your way inward, two at a time. Every pair has a sum of 101.

$$\begin{array}{rrrrcr} 100 & 99 & 98 & 97 & & 51 \\ \underline{+1} & \underline{+2} & \underline{+3} & \underline{+4} & \ldots\ldots & \underline{+50} \\ 101 & 101 & 101 & 101 & & 101 \end{array}$$

Since the number of pairs is half of 100, the sum of 1 through 100 has to be $\frac{100}{2}\cdot 101$ or 5,050. Since $\frac{n}{2}$ is the same as $\frac{100}{2}$ and a_1+a_n equals 101, this is just the formula $\frac{n}{2}(a_1+a_n)$ applied to the specific sequence of the first 100 positive integers. Gauss had figured all of this out in his head in just a few seconds (without anyone ever teaching him about sequences)!

There's also a formula for calculating the sum of any finite geometric sequence. If $\{a_1, a_2, a_3, ..., a_n\}$ is a finite geometric sequence, then the sum can be calculated with the formula below.

$$\frac{a_1(1-r^n)}{1-r} \quad \textbf{sum of geometric sequence}$$

Remember, r is the common ratio of the sequence and n is the number of terms that are being added. Let's use this formula to calculate the sum of the first 5 terms of the geometric sequence $3\cdot 2^{n-1}$. Here are the first 5 terms.

3, 6, 12, 24, 48

The terms add to 93. Applying the formula, we get the following.

$$\frac{a_1(1-r^n)}{1-r} = \frac{3(1-2^5)}{1-2} = \frac{3(-31)}{-1} = 93$$

It works. A couple of other examples are shown below.

sum of first 4 terms of 2^n = 2+4+8+16

$$\frac{a_1(1-r^n)}{1-r} = \frac{2(1-2^4)}{1-2} = \frac{2(-15)}{-1} = 30$$

sum of first 5 terms of $(\frac{1}{3})^n$ = $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\frac{1}{243}$

$$\frac{a_1(1-r^n)}{1-r} = \frac{\frac{1}{3}(1-\frac{1}{3}^5)}{1-\frac{1}{3}} = \frac{\frac{1}{3}(1-\frac{1}{243})}{\frac{2}{3}} = \frac{121}{243}$$

[1] The story may or may not be true.

Sum of an Infinite Sequence

When calculating the sum of a sequence, it's common sense that the sequence would have to be finite. After all, how could you add up an infinite number of terms? The answer would obviously be infinite. For instance, look how fast the sums of the geometric sequence 3^n increase as we include more and more terms.

$$3+3^2+3^3+3^4=120$$

$$3+3^2+3^3+3^4+3^5=363$$

$$3+3^2+3^3+3^4+3^5+3^6=1,092$$

$$3+3^2+3^3+3^4+3^5+3^6+3^7=3,279$$

$$3+3^2+3^3+3^4+3^5+3^6+3^7+3^8=9,840$$

$$3+3^2+3^3+3^4+3^5+3^6+3^7+3^8+3^9=29,523$$

The sum of this sequence is obviously headed toward infinity. By the way, when we add just some of the terms of a sequence, the answer is called a **partial sum**. So 120, 363, 1,092 and the other answers above are partial sums of the sequence 3^n. The point is that the sum of an infinite sequence doesn't seem to have common sense.

But mathematics isn't based on common sense; it's based on logic and proof. And believe-it-or-not, there are some infinite sequences that actually can be added up to get a numerical answer. Here's an example of one.

$$3+\frac{3}{2}+\frac{3}{4}+\frac{3}{8}+\frac{3}{16}+\frac{3}{32}+,...$$

This is the sum of the infinite geometric sequence, $3\left(\frac{1}{2}\right)^{n-1}$. Now look at several of the partial sums.

$$3+\frac{3}{2}+\frac{3}{4}=\frac{21}{4} \text{ or } 5\frac{1}{4}$$

$$3+\frac{3}{2}+\frac{3}{4}+\frac{3}{8}=\frac{45}{8} \text{ or } 5\frac{5}{8}$$

$$3+\frac{3}{2}+\frac{3}{4}+\frac{3}{8}+\frac{3}{16}=\frac{93}{16} \text{ or } 5\frac{13}{16}$$

$$3+\frac{3}{2}+\frac{3}{4}+\frac{3}{8}+\frac{3}{16}+\frac{3}{32}=\frac{189}{32} \text{ or } 5\frac{29}{32}$$

$$3+\frac{3}{2}+\frac{3}{4}+\frac{3}{8}+\frac{3}{16}+\frac{3}{32}+\frac{3}{64}=\frac{381}{64} \text{ or } 5\frac{61}{64}$$

$$3+\frac{3}{2}+\frac{3}{4}+\frac{3}{8}+\frac{3}{16}+\frac{3}{32}+\frac{3}{64}+\frac{3}{128}=\frac{765}{128} \text{ or } 5\frac{125}{128}$$

If we were to continue this process, adding more and more terms, the partial sums would get closer and closer to 6, but never quite reach that number. In fact, since this is a geometric sequence we can calculate some partial sums with larger numbers of terms. Remember, the formula $\frac{a_1(1-r^n)}{1-r}$ can be used to calculate the sum of a finite geometric sequence. In this case, $a_1 = 3$ and $r = \frac{1}{2}$. That gives us this.

$$\frac{3(1-\frac{1}{2}^{n})}{1-\frac{1}{2}}$$

Now let's calculate the partial sum when $n = 50$.

$$\frac{3(1-\frac{1}{2}^{50})}{1-\frac{1}{2}} = \frac{3(1-0.00000000000000089)}{\frac{1}{2}}$$

Notice how small is the number $\left(\frac{1}{2}\right)^{50}$. That makes the quantity inside the parentheses just a tiny bit less than 1.

$$\frac{3(0.99999999999999911)}{\frac{1}{2}}$$

Multiplying by 3 gives a product on top that's ever so slightly less than 3.

$$\frac{2.9999999999999973}{\frac{1}{2}}$$

The last step is to invert the $\frac{1}{2}$ and multiply.

$$(2.9999999999999973)(2)$$

$$5.99999999999999946$$

The answer is just a little under 6.

Are you starting to get the idea here? No matter how many terms of the infinite sequence $3\left(\frac{1}{2}\right)^{n-1}$ we add, the sum is always under 6. And the more terms we add, the closer the sum gets to 6. It never quite reaches 6 exactly, though. The technical word for this kind of situation is "limit." We say that as *n* approaches infinity, the expression approaches a limit of 6. It's basically the same concept as a function approaching an asymptote. The graph gets closer and closer to some line, but never crosses it. We'll talk more about limits at the end of this course. Limits are of the very greatest importance in calculus. For now, the main point is that since the partial sums of the sequence $3\left(\frac{1}{2}\right)^{n-1}$ get closer and closer to 6, mathematicians decided to define the sum of the infinite sequence (where *n* keeps going to infinity) to equal 6 exactly. That's how it's possible for some infinite sequences to have sums. It all depends on whether their partial sums, as the terms increase, approach some limit.

There are some other technical words related to this topic that you should know. An expression containing an infinite number of terms added together is called an **infinite series**. So the infinite sequence of $3\left(\frac{1}{2}\right)^{n-1}$ qualifies as an infinite series. In general form, we write an infinite series like this.

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + ... + a_n + ...$$

Notice that the infinity symbol is placed on top of sigma to show that *n* goes all the way to infinity. An infinite series that approaches a limit (and actually has a sum) is said to "converge." An infinite series, such as the one for 3^n, that does not approach a limit is said to "diverge." There are tests that can be applied to determine whether a particular series converges or diverges, but they usually require more advanced math knowledge.

There's one convergence test, however, that you can learn right now. It's the test for any geometric series. As it turns out, a geometric series will always converge if its common ratio (r) is between -1 and 1. If it's outside that range, then the series will diverge (and not have a sum).

Convergence test for geometric series

$$-1 < \text{common ratio} < 1$$

The series for $3\left(\frac{1}{2}\right)^{n-1}$ converged to 6, because the common ratio was $\frac{1}{2}$, which is between -1 and 1. However, the geometric series $3+3^2+3^3+3^4+...$ does not converge, because it's common ratio is 3. That's outside the range of $-1 < r < 1$. As we saw above, the partial sums of $3+3^2+3^3+3^4+...$ grow faster and faster, without approaching any limit. This convergence test for any geometric series makes a lot of sense. The test is really saying that the common ratio has to be a proper fraction in order for the series to approach a limit. When the common ratio is a proper fraction, each term will be smaller than the previous one by that fraction. If the common ratio is $\frac{1}{2}$, then each term will be one-half of the previous term. If the common ratio is $\frac{1}{3}$, each term will be one-third of the previous term. When the common ratio is a proper fraction, the terms get smaller and smaller. So it makes sense that the sum could approach a limit. When the common ratio is greater than 1 or less than -1, the magnitude of each term will be bigger than the previous one.[2]

Practice 78

a. Write a direct formula (n^{th} term) for the arithmetic sequence 5, 13, 21, 29, …?

A. 8^n B. 8^{n-1} C. $8n$
D. $8n-8$ E. $8n-3$

b. Find the sum of the first 70 terms of the arithmetic sequence $7+16+25+...+628$.

[2] The tests for convergence are very complex. It's not enough for each term to be smaller than the previous one. Some series with smaller and smaller terms still don't converge. A knowledge of calculus is required to determine whether most series actually converge.

c. Use the convergence test to determine whether the geometric sum $12+(-3)+\frac{3}{4}+\left(-\frac{3}{16}\right)+\ ...$ converges or diverges?

d. Name a region of the *xy* plane whose points satisfy either the inequality $y<-x^2$ or the inequality $y<0$.

A. a square
B. the half of the plane below the *x*-axis
C. a circle
D. the region of the plane bounded by a parabola
E. a crescent-shaped region in the plane

e. Solve the word problem below.

The dividend paid on a stock have increased at rate of 2% since 1996. The value of the dividend can be represented by the function $C(t)=500(1.02)^t$ where *t* is time in years after 1996. Based on this, what is the total amount one would have earned from the dividends in 2004 if one had held the stock since 1997? Estimate your answer to the nearest cents.

Problem Set 78

Tell whether each sentence below is True or False.

1. To calculate the sum of any arithmetic sequence, add the first and last term of the sequence then multiply that total by the number of terms divided by 2.

2. The formula for calculating the sum of any finite geometric sequence is $\frac{n}{2}(a_1+a_n)$.

For each pair of rectangular coordinates below, select the correct conversion into polar coordinates. Don't use the polar conversion function on your calculator.

3. $G(21,48)$

A. $(52.4, 66.4°)$
B. $(-52.4, 113.6°)$
C. $(-52.4, 66.4°)$
D. $(69, -66.4°)$
E. $(52.4, -66.4°)$

4. $H(-52,75)$

A. $(-91.3, -124.7°)$
B. $(91.3, -124.7°)$
C. $(91.3, 124.7°)$
D. $(-91.3, 124.7°)$
E. $(23, -55.3°)$

Select the standard form for each conic section equation below.

5. $y^2-2y+3x+10=0$

A. $x+3=-3(y-1)^2$
B. $x+3=-\frac{1}{3}(y-1)^2$
C. $y-\frac{1}{3}=3(x-1)^2$
D. $x+\frac{1}{3}=-(y-1)^2$
E. $x-3=3(y-1)^2$

6. $9x^2 - 54x + 16y^2 + 64y = -1$

A. $\dfrac{(x-3)^2}{3^2} + \dfrac{(y+2)^2}{4^2} = 1$ B. $y - 16 = 9(x-3)^2$ C. $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$

D. $16(x+3)^2 + 9(y-2)^2 = 144$ E. $\dfrac{(x-3)^2}{4^2} + \dfrac{(y+2)^2}{3^2} = 1$

Find the next three terms in each sequence below.

7. 40, 37, 34, 31 … **8.** 4, 8, 16, 32 …

Select the correct answer for each sequence below.

(a) 9. Write a direct formula (n^{th} term) for the arithmetic sequence 8, 13, 18, 23, …?

A. $5n$ B. 5^n C. $5n+3$

D. $5n-5$ E. 5^{n-1}

10. Write a recursive formula for the geometric sequence 4, -2, 1, $-\dfrac{1}{2}$, …?

A. $a_n = a_{n-1} - 6$ B. $a_n = \left(-\dfrac{1}{2}\right)^n$ C. $a_n = -\dfrac{1}{2}n$

D. $a_n = -\dfrac{1}{2}a_{n-1}$ E. $a_n = \dfrac{1}{2}a_{n-1}$

Answer each question below.

11. Find the 14^{th} term in the arithmetic sequence with $a_1 = 3$ and a common difference (d) of -6.

12. Find the common ratio (r) in the geometric sequence: $6,\ 42,\ 294,\ 2{,}058$ ...

13. Find the common difference (d) for the arithmetic sequence with $a_1 = 35$ and $a_{12} = 68$.

14. Find the sum of the whole numbers from 1 to 50.

(b) 15. Find the sum of the first 60 terms of the arithmetic sequence $8 + 17 + 26 + ... + 539$.

Use the convergence test to determine whether each geometric sum below converges or diverges?

(c) 16. $6 + (-2) + \dfrac{2}{3} + \left(-\dfrac{2}{9}\right) + \ ...$ **17.** $5 + 10 + 20 + 40 + \ ...$

Select the answer for each question below.

(d) 18. Name a region of the xy plane whose points satisfy either the inequality $y > 2x^2$ or the inequality $y > 0$.

A. a circle
B. a crescent-shaped region in the plane
C. the half of the plane above the x-axis
D. the region of the plane bounded by a parabola
E. a square

19. Some number n is added to the three numbers, -3, 7 and 67, to create the first three terms of a geometric sequence. What is the value of n?

A. 0
B. 1
C. 3
D. 4
E. 5

Answer the question below.

20. If $f(x) = \sqrt[3]{27x-5}$, then $f(f^{-1}(3)) = ?$ Estimate your answer to a whole number

21. A line has parametric equations $x = t-9$ and $y = 5t-3$, given t is the parameter. What is the y-intercept of the line that represents the direct relationship between x and y?

Solve the word problem below.

(f) 22. Annual insurance premiums have increased at a rate of 10% since 1990. The annual cost of insurance can be represented by the function $C(t) = 1{,}000(1.1)^t$ where t is time in years after 1990. Based on this, what is the total amount one would have paid by 2002 to the insurance company if one had been a member since 1991? Estimate your answer to the nearest cents.

Lesson 79—More on Infinite Series

In the last lesson, we learned that some infinite series have a finite sum, which is really surprising. Those series are said to "converge," and the others that don't have a finite sum "diverge." And remember, any geometric series will converge if its common ratio is between -1 and 1.

It's nice to be able to tell whether a particular geometric series converges. But it's nicer still to be able to calculate quickly the actual sum of a converging series. In the last lesson, we did that with the geometric series for $3\left(\frac{1}{2}\right)^{n-1}$. We used the formula $\frac{a_1(1-r^n)}{1-r}$. After plugging in the values for $3\left(\frac{1}{2}\right)^{n-1}$, the formula became $\frac{3(1-\frac{1}{2}^n)}{1-\frac{1}{2}}$. Remember, as n increased, $\frac{3(1-\frac{1}{2}^n)}{1-\frac{1}{2}}$ got closer and closer to 6.

Formula for an Infinite Geometric Series

We could have come up with 6 a lot easier by just analyzing the expression $\frac{3(1-\frac{1}{2}^n)}{1-\frac{1}{2}}$. As n approaches infinity, the term $\left(\frac{1}{2}\right)^n$ gets smaller and smaller. To be more accurate, it approaches a limit of 0. That means, to calculate the limit of $\frac{3(1-\frac{1}{2}^n)}{1-\frac{1}{2}}$ as n approached infinity, all we have to do is replace $\left(\frac{1}{2}\right)^n$ with 0. That gives us this.

When n approaches infinity

$$\frac{3(1-\frac{1}{2}^n)}{1-\frac{1}{2}} \longrightarrow \frac{3(1-0)}{1-\frac{1}{2}}$$

Now we can simplify normally to get the limit of 6.

$$\frac{3}{\frac{1}{2}}=6$$

The same kind of analysis can be performed on any other geometric series with a common ratio between -1 and 1. We can get the limits for those as well.

Fortunately, there's an even easier way to calculate limits for converging geometric series. We can take the limit of the general formula $\frac{a_1(1-r^n)}{1-r}$. When $-1<r<1$, we know that r^n will approach 0 as x approaches infinity. So we can just replace r^n with 0 and simplify.

$$\frac{a_1(1-0)}{1-r} \longrightarrow \frac{a_1}{1-r}$$

The formula $\frac{a_1}{1-r}$ can be used to immediately calculate the limit of any converging geometric series. The series below has an initial term of 5 and a common ratio of $\frac{1}{3}$. So $a_1=5$ and $r=\frac{1}{3}$.

$$5+\frac{5}{3}+\frac{5}{9}+\frac{5}{27}+\frac{5}{81}+\ldots$$

Plugging $a_1 = 5$ and $r = \frac{1}{3}$ into the formula gives us a limit of 7.5.

$$\frac{a_1}{1-r} \longrightarrow \frac{5}{1-\frac{1}{3}} \longrightarrow 7.5$$

That means if we were to keep adding the terms of the series $5+\frac{5}{3}+\frac{5}{9}+\frac{5}{27}+\frac{5}{81}+...$ forever, the partial sums would get closer and closer to 7.5 but never quite reach it. When using the formula $\frac{a_1}{1-r}$, just remember that it only works for geometric series (not other kinds). And it only works when the common ratio is between -1 and 1.

It's kind of interesting that a repeating decimal is really just an infinite geometric series. Take 0.3333…, for example. This means $\frac{3}{10}+\frac{3}{100}+\frac{3}{1,000}+\frac{1}{10,000}+...$, which is geometric, and has a common ratio of $\frac{1}{10}$. And, of course, the fractions just keep going forever. When students are first taught that $\frac{1}{3}$ is the same as 0.3333…, they're too young to appreciate how strange it is that an infinite number of fractions could add up to equal the finite sum $\frac{1}{3}$. They're just taught to memorize that $\frac{1}{3}$ and 0.3333… are the same. But now you know that some infinite series' do have finite sums. When such a series is geometric, the sum can be calculated with the formula $\frac{a_1}{1-r}$. Let's prove that $\frac{3}{10}+\frac{3}{100}+\frac{3}{1,000}+\frac{1}{10,000}+...$ does actually equal $\frac{1}{3}$. The first term is $\frac{3}{10}$. So we need to put that in for a_1. And the common ratio is $\frac{1}{10}$, so that should go in for *r*.

$$\frac{a_1}{1-r} \longrightarrow \frac{\frac{3}{10}}{1-\frac{1}{10}}$$

Simplifying gives us this.

$$\frac{\frac{3}{10}}{\frac{9}{10}} \longrightarrow \frac{3}{9} \longrightarrow \frac{1}{3}$$

It works.

All repeating decimals are really infinite geometric series'. And since our number system is based on 10s, they all have a common ratio of $\frac{1}{10}^n$. So we can use the formula $\frac{a_1}{1-r}$ (with $r = \frac{1}{10}^n$) to convert any repeating decimal into a fraction.

Mathematical Induction

There's a method of doing mathematical proofs that's particularly useful when working with sequences of positive integers. The method is called mathematical induction. To show you how it works, let's say that we want to prove that the sum of n odd positive integers always equals n^2.[3]

$$1+3+5+7+...+2n-1=n^2$$

We could try to prove this by checking for several values of n. For instance, if $n=3$, the rule works.

$$1+3+5=3^2$$

$$9=9$$

The rule also works for $n=5$.

$$1+3+5+7+9=5^2$$

$$25=25$$

The problem with checking for specific values is that it doesn't prove that n^2 works for all possible values for n. That's much tougher to prove, because n could be anything, even some absolutely huge number. As you may remember from geometry, showing that a rule is true for several individual cases doesn't qualify as a mathematical (deductive) proof.

But we can do a proper proof using mathematical induction. The first step is to show that n^2 is right for just the initial term of the sequence—for $n=1$, in other words. Not surprisingly, it works.

$$1=1^2$$

$$1=1$$

Next, we *assume* that n^2 is right for some other value for n. We'll call it k. Using the expression $2n-1$, the k^{th} term of the sequence must be $2k-1$. That gives us the equation below.

$$1+3+5+7+...+(2k-1)=k^2$$

Assuming that this equation is true, let's now put in the next term on the left. The term following $(2k-1)$ can be obtained by substituting $k+1$ for k in the expression $2n-1$. That gives us $2(k+1)-1$. Of course, if we add a quantity to the left side of an equation, we have to add the same quantity to the right side.

$$1+3+5+7+...+(2k-1)+2(k+1)-1=k^2+2(k+1)-1$$

Next, we simplify the right side.

$$1+3+5+7+...+(2k-1)+2(k+1)-1=k^2+2k+1$$

[3] The $2n-1$ is just the formula for generating any particular term of this sequence. Putting 1 in for n makes $2n-1$ equal 1, which is the first odd integer. Putting 2 in for n makes $2n-1$ equal 3, which is the second odd integer, and so on.

The equation now says that the series through the $k+1^{\text{th}}$ term is equal to k^2+2k+1. If that's really true, then we should get k^2+2k+1 when we substitute $k+1$ for *n* in the expression n^2. That gives us $(k+1)^2$, which simplifies to equal $k^2+k+k+1$ or k^2+2k+1, so it works. That finishes the proof.

Now let's think about what we just did. We proved that our rule works for $n=1$, which is the very first term of the sequence. Then we proved that *if* the rule works for some value $n=k$, then it must also work for $n=k+1$, which is the sequence with the very next term added. That word *if* is important, because we just assumed that the rule worked for *k*, remember. But what if $k=1$? We know the rule works for 1. But we've already proved that it also must work for the next value for *n* which is 2. But then we can let $k=2$. And we've proved that if the rule works for 2, then it also must work for 3. Since we've proved that the rule works for the first term (1) and that it will always work for the very next value for *n*, then we have effectively proven that the rule works for any *n* all the way up to infinity. That's how mathematical induction can be used to prove things that are true for all positive integers.

Practice 79

a. If the common ratio (*r*) is 3 and $a_6=162$, find the first term of the geometric sequence.

b. Find the sum of the infinite series $1+\frac{3}{4}+\frac{9}{16}+\frac{27}{64}\ ...$, or state that the sum does not exist and explain your reasoning.

c. Convert the repeating decimal $0.8888...$ into a fraction.

d. Use mathematical induction to prove that the proposition $1+2+4+...+2^{n-1}=2^n-1$ is valid for all positive integral values of *n*.

e. If the 19$^{\text{th}}$ term of an arithmetic sequence is 195, and the 27$^{\text{th}}$ term is 275, then what is the first term of the sequence?

f. Solve the word problem below.

During a physical exercise drill called a suicide, athletes have to run to the first marked line then back to the starting point. They then turn around and run to the next marked line and then back again to the starting point. This repeats until the athlete reaches the final marked line. The distance in feet ran for each line can be represented by the function $D(l)=2(15l)$ where *l* is the number of the line the athlete ran to and from. How many feet would an athlete have run total if he had just finished running to and from the 6$^{\text{th}}$ line?

Problem Set 79

Tell whether each sentence below is True or False.

1. It is impossible to calculate the sum of an infinite series.

2. Mathematical induction is a method for doing proofs that's particularly useful when working with sequences of positive integers.

Select the equation for each conic section below.

3. The center of an ellipse is at $(3,-5)$, the length of the major axis is 14 units, and the length of the minor axis is 6 units.

A. $\frac{(x+5)^2}{49}+\frac{(y-3)^2}{9}=1$ B. $\frac{x^2}{196}+\frac{y^2}{36}=1$ C. $\frac{(x-3)^2}{28^2}+\frac{(y+5)^2}{12^2}=1$

D. $\frac{(x-3)^2}{7^2}+\frac{(y+5)^2}{3^2}=1$ E. $\frac{(x-3)^2}{3^2}-\frac{(y+5)^2}{7^2}=1$

4. Select the center and vertices of the hyperbola $\frac{y^2}{64}-\frac{x^2}{49}=1$.

A. Center $(0,0)$; Vertices $(0,-7)$, $(0,7)$ B. Center $(0,7)$; Vertices $(0,8)$, $(8,0)$

C. Center $(8,0)$; Vertices $(-7,0)$, $(7,0)$ D. Center $(0,0)$; Vertices $(-8,0)$, $(8,0)$

E. Center $(0,0)$; Vertices $(0,-8)$, $(0,8)$

Select the correct answer for each sequence below.

5. Write a recursive formula for the arithmetic sequence 5.5, 7, 8.5, 10, …?

A. $a_n=1.5^n$ B. $a_n=a_{n-1}+1.5$ C. $a_n=2a_{n-1}-4$

D. $a_n=1.5n$ E. $a_n=1.5a_{n-1}$

6. Write a direct formula (n^{th} term) for the geometric sequence $243,\ -81,\ 27,\ -9\ldots$?

A. $243\left(-\frac{1}{3}\right)^{n-1}$ B. $-\frac{1}{3}n$ C. $\left(-\frac{1}{3}\right)^n$

D. $n-\frac{1}{3}$ E. $243\left(\frac{1}{3}\right)^n$

Answer each question below.

7. Find the 20^{th} term in the arithmetic sequence with $a_1=2$ and a common difference (d) of 13.

8. Find the common ratio (r) in the following geometric sequence: $\frac{1}{2},\ -2,\ 8,\ -32,\ 128\ \ldots$

(a) 9. If the common ratio (r) of a geometric sequence is 2 and $a_5=12$, find the first term of the sequence.

10. Find the sum of the first 40 terms of the arithmetic sequence $6+13+20+\ldots+279$.

Find the sum of each infinite geometric series, or indicate that the sum does not exist.

11. $20+10+5+2.5+\ ...$

(b) 12. $2+1+\frac{1}{2}+\frac{1}{4}+\ ...$

Convert each repeating decimal below into a fraction.

13. $0.2222...$

(c) 14. $0.7777...$

Use mathematical induction to prove that the equation below is true for all positive integer values of *n*.

(d) 15. $1+2+3+...+n=\frac{n(n+1)}{2}$

Select the answer for each question below.

16. Point $(-6,4)$ is located on a circle in the coordinate plane given by the equation $(x+2)^2+(y-1)^2=r^2$. Which of the following points must also lie on the circle?

A. $(-3,-2)$
B. $(1,-3)$
C. $(2,2.5)$
D. $(-7,4)$
E. $(1.2,-2)$

17. In an arithmetic sequence, $a_6=a_{11}-10$ and $a_4=-3$. Between which two consecutive terms does 0 lie?

A. a_4 and a_5
B. a_5 and a_6
C. a_6 and a_7
D. a_7 and a_8
E. a_9 and a_{10}

18. In a geometric series, suppose $g_2=4$ and $g_5=108$. What is $\frac{g_n}{g_{n-1}}=?$

A. $\frac{1}{3}$
B. 2
C. 3
D. 4
E. 8

Answer the question below.

19. What is the remainder when the polynomial $x^4-9x^2+20x-13$ is divided by $x+3$?

20. What is the maximum value of $f(x)=4-(x-6)^2$?

(e) 21. If the 17th term of an arithmetic sequence is 180, and the 29th term is 300, then what is the first term of the sequence?

Solve the word problem below.

(f) 22. During a physical exercise drill called a suicide, athletes have to run to the first marked line then back to the starting point. They then turn around and run to the next marked line and then back again to the starting point. This repeats until the athlete reaches the final marked line. The distance in feet ran for each line can be represented by the function $D(l)=2(20l)$ where *l* is the number of the line the athlete ran to and from. How many feet would an athlete have run total if he had just finished running to and from the 10th line?

Lesson 80—Probability

The word "probability" is used a lot in day-to-day conversation. People will say that the probability of rain is 60% or the probability of a candidate winning an election is 75%. Basically, probability is a measure of how certain something is to happen in the future is. Probabilities can be written as percentages, fractions, or decimals. So the probabilities 60% and 75% can also be written as 0.60 and 0.75 or as $\frac{3}{5}$ and $\frac{3}{4}$ (which are the fractions $\frac{60}{100}$ and $\frac{75}{100}$ fully reduced). When an event is absolutely certain to happen, the probability is 100% or 1. When an event will definitely not happen, the probability is 0% or just 0. So all probabilities range from 0 to 1.

Calculating Probability

How is probability actually calculated? Well, the first step is to count all of the possible ways that something can happen. That's the total number of possible "outcomes." As an example, imagine flipping a coin. There are two possibilities. The coin can land heads or tails. The total number of possible outcomes is therefore equal to 2. That number goes in the bottom of a fraction. Now let's say we want to calculate the probability that the coin will land heads. Heads is the favorable outcome. Since there's just one way the coin can land heads, the number of favorable outcomes equals 1, which goes in the top of the fraction.

$$\frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes}} = \frac{1}{2}$$

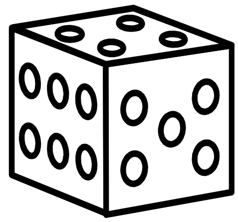
The fraction equals $\frac{1}{2}$, which means the probability of the coin landing heads is $\frac{1}{2}$ or 0.5 or 50%, depending on how you want to write it. That's how to calculate probability. It's the number of favorable outcomes (what you're calculating the probability of) divided by the total number of outcomes.

Table 80.1

Calculating probability	$\text{Probability} = \dfrac{\text{Number of favorable outcomes}}{\text{Total number of outcomes}}$

This doesn't mean that a coin is guaranteed to land heads one time out of every two flips. Probabilities aren't that certain. A probability of $\frac{1}{2}$ means that if you flip a coin many, many times, the number of heads will be approximately equal to $\frac{1}{2}$ of all the flips. Also, the method of calculating probabilities (dividing favorable outcomes by total outcomes) only works when the possibilities are all equally likely. What if an Olympic swimmer was considering swimming in the shallow end of a pool? There are two possibilities, he could drown or not drown. Does that mean that he has a $\frac{1}{2}$ or 50% probability of drowning? No, of course not. The analysis is wrong, because the two possibilities aren't equally likely. An expert swimmer is extremely unlikely to drown in the shallow end of a pool. The probability might be $\frac{1}{100{,}000}$. Since a coin is symmetrical in shape, we assume that the coin is just as likely to land heads as tails. That's why we can calculate the probability by dividing favorable outcomes by total possible outcomes.

As another example, let's calculate the probability of rolling an even number with a single die.



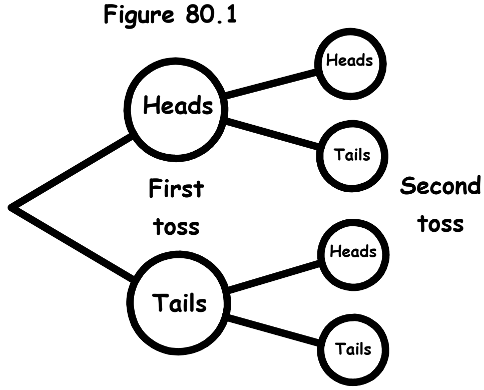
There are 6 possible outcomes: 1, 2, 3, 4, 5, and 6. So 6 goes in the bottom of the fraction. The favorable outcomes are the even numbers, which are 2, 4, and 6. That's 3 favorable outcomes. So we need to put 3 in the top of the fraction.

$$\frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes}} = \frac{3}{6} = \frac{1}{2}$$

That gives us $\frac{3}{6}$ or $\frac{1}{2}$, which means that the probability of rolling an even number with a die is also equal to $\frac{1}{2}$.

Sometimes we need to calculate the probability of two events happening. For instance, we might want to calculate the probability of flipping a coin heads two times in a row. In cases like this, figuring out the total number of outcomes is a little harder. One way to do it is to draw a tree diagram. On the first flip, there are two possibilities, so we draw two branches of the "tree." If the coin lands heads, then we will flip it a second time, and there will be two possibilities again. So we draw two branches extending out from the first heads. If the coin lands tails on the first flip, then we will also flip a second time. So we draw two more branches extending out from the first tails.

Figure 80.1



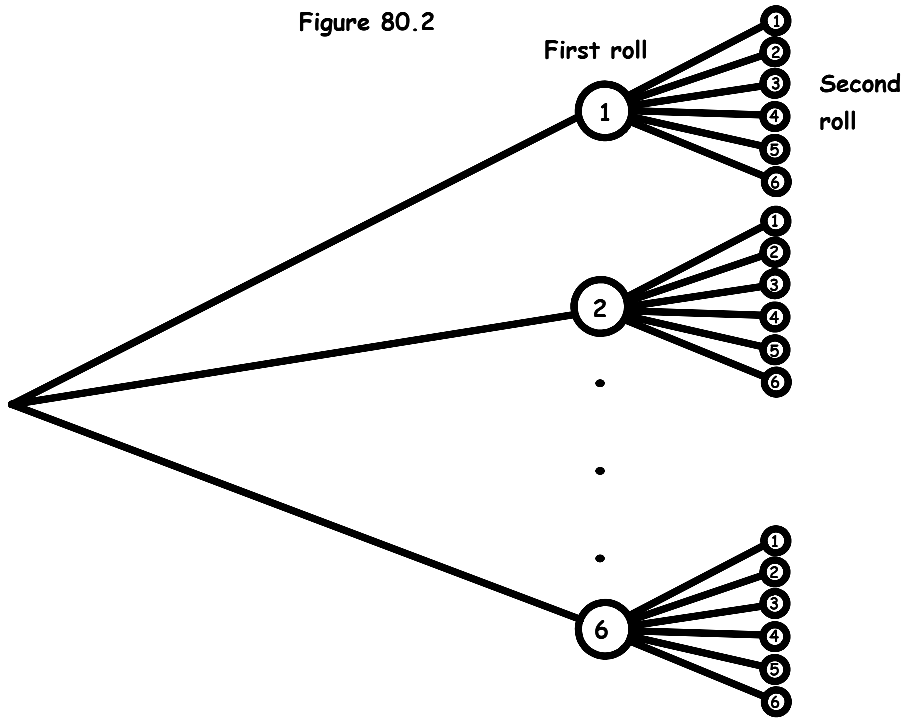
To find the total number of possible outcomes, we just have to count all the branches on the far right of the diagram. There are 4 branches, which means flipping a coin twice has 4 possible outcomes. We need to put 4 in the bottom of the fraction. To get the favorable outcomes, we just need to find how many branches involve two heads. There's just one. The branch on top. So 1 goes on top of the fraction.

$$\frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes}} = \frac{1}{4}$$

The probability of flipping a coin twice and having it land heads both times is $\frac{1}{4}$ or 25%.

A tree diagram is even more helpful in calculating the probability of rolling a 10 with two dice. There are 6 possibilities when rolling the first die. The die could land 1, 2, 3, 4, 5, or 6. So that's 6 branches to start. Whichever one of the cases occurs, we will roll the second die. And there will be 6 possibilities for it. So we need to draw 6 branches extending from each of the 6 original possibilities.

Figure 80.2



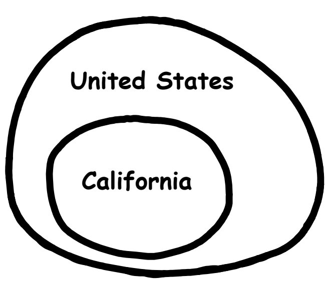
To get the total number of possible outcomes, we just count all the possibilities on the far right. There are 36. So 36 goes in the bottom of the fraction. How do we find the favorable outcomes? We just count the number of outcomes on the right where the dice add to 10. There's 4 and 6, 5, and 5, 6, and 4. That's 3 favorable outcomes, so 3 goes in the top of the fraction.

$$\frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes}} = \frac{3}{36} = \frac{1}{12}$$

After reducing, we end up with a probability of $\frac{1}{12}$ or about 8.3%. That's pretty low.

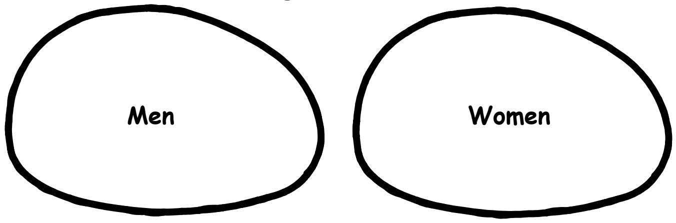
Venn diagrams, which you may remember from geometry, can also be helpful when calculating probabilities. Venn diagrams are just circles (or other shapes) that represent a group or "set" of objects (or elements). Putting one circle inside the other means that the inner circle is a "subset" of the big one. For example, people living in California are a subset of people living in the U.S.

Figure 80.3



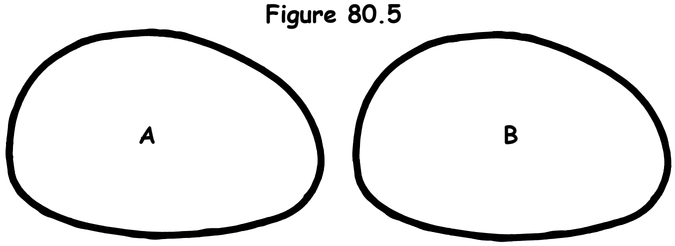
If two sets have no members in common, then the circles are entirely separate. The set of women and the set of men are an example.

Figure 80.4



Sometimes, in figuring out a probability, we might need to include all the objects (elements) in a set *A* as well as all of the objects (elements) in another set *B*. This is called the union of *A* and *B* and it's written like this: $A \cup B$.

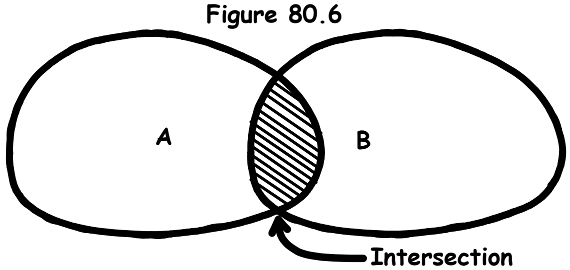
Figure 80.5



The union of A and B equals A+B

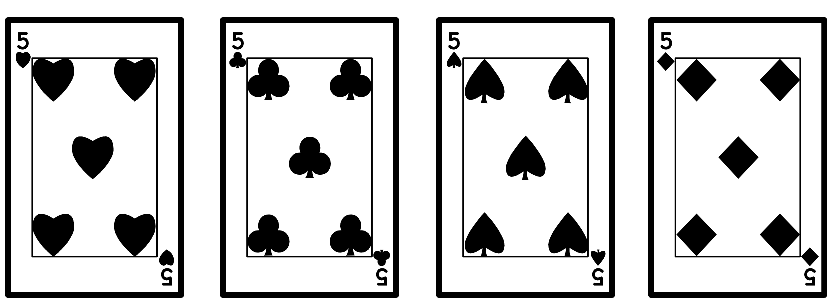
When two sets partially overlap, the sets are said to "intersect." The intersection includes all the objects that are members of both sets. The intersection of two sets *A* and *B* is drawn by overlapping the circles (see below). The symbol is $A \cap B$.

Figure 80.6

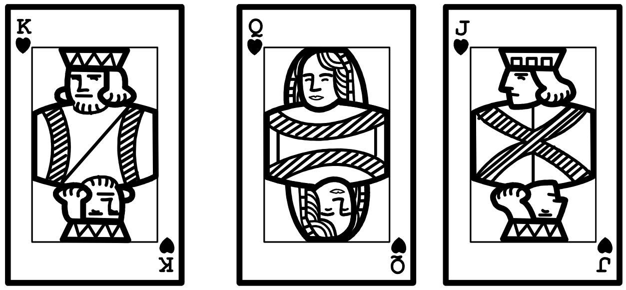


The intersection of A and B
equals the objects in both sets

Here's how Venn diagrams might be used to calculate a probability. What if we wanted to calculate the probability of drawing a diamond or a face card from a deck of cards. Just in case you're not that familiar with cards, a card deck has 52 cards in total. There are 4 types of cards: diamonds, hearts, clubs, and spades.

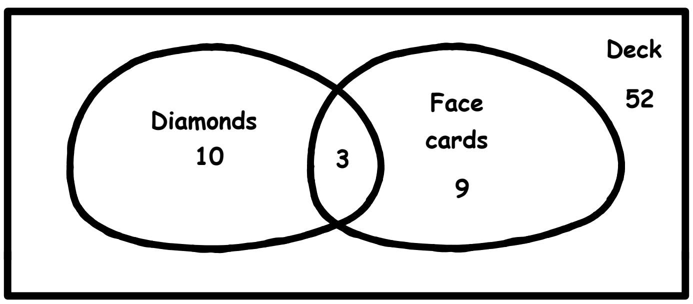


Each type represents $\frac{1}{4}$ of the deck, which is 13 cards each. A face card is a card that has a face on it. There are three types of face cards: jacks, queens, and kings.



There are 3 face cards in each of the 4 categories of cards (diamonds, hearts, clubs, and spades). The first step in calculating the probability of drawing a diamond or a face card is to figure out the total possible outcomes. That's just 52, since there are 52 cards in total. We'll draw a rectangle to represent the entire deck of cards. The diamonds are a subset of the entire deck and so are all the face cards. Importantly, though, the set of diamonds intersects with the set of face cards. That's because 3 of the face cards are also diamonds.

Figure 80.7



To figure out the total favorable outcomes, we need to add the number of diamonds to the number of face cards and then subtract the intersection between the two. The subtraction prevents us from double-counting the 3 diamond face cards (that are in both sets). There are 13 diamonds, 12 face cards (3 for each of the 4 categories of cards). The intersection between the two sets is 3. So we get $13+12-3=22$. The number of favorable outcomes, then, is 22. The probability of drawing a diamond or a face card is $\frac{22}{52}$ or $\frac{11}{26}$, which is about 42.3%.

$$\frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes}}=\frac{22}{52}=\frac{11}{26}$$

Multiplication Principle of Counting

It can be pretty tedious to draw tree diagrams every time you want to count the total outcomes or the favorable outcomes in a situation. That's why there are counting shortcuts that can be used instead. When a probability calculation involves multiple events (like two coin flips or two dice), the rule is that the total number of outcomes is equal to the product of the possible outcomes of each event. If the total number of outcomes of event *A* is symbolized as n(A) and the total number of outcomes of event *B* is symbolized as n(B), then the following is true.

Table 80.2

Multiplication Principle of Counting	**n(A and then B)=n(A).n(B)**

Using the multiplication principle of counting on flipping a coin twice, we just take the 2 possibilities of the first flip times the 2 possibilities of the second flip to get a total number of outcomes of 4, which we know is right.

1^{st} flip		2^{nd} flip		
2 outcomes	x	2 outcomes	=	4 total outcomes

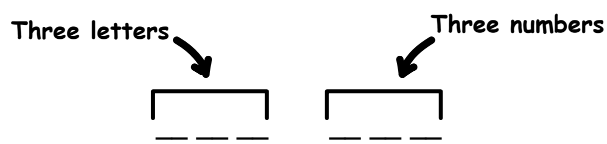
Similarly, we can use the shortcut on rolling dice. The first roll has 6 possible outcomes and the second roll also has 6 possible outcomes. Since 6 times 6 equals 36, there must be 36 possible outcomes for both rolls. This is also correct.

1^{st} roll		2^{nd} roll		
6 outcomes	x	6 outcomes	=	36 total outcomes

The multiplication principle of counting can be applied no matter how many events are involved. Here's a tougher example.

> Sam wants to password his personal webpage. If he uses pass-words with 6 characters, 3 letters followed by 3 numbers, how many possible passwords can be created?

To do this problem, the first step is to imagine 6 blank spaces where the letters and numbers go.



Since the alphabet has 26 letters, there are 26 possibilities that could go in the first letter blank, 26 in the second, and 26 in the third. Since our numbers have 10 possible digits, there are 10 possibilities that could go in the first number blank, 10 in the second, and 10 in the third. Using the multiplication principle of counting, we get the following.

$$26 \times 26 \times 26 \times 10 \times 10 \times 10 = 17,576,000$$

Believe-it-or-not there are 17,576,000 different 3 letter-3 number passwords that can be created. In a problem like this, drawing a tree diagram to count all the outcomes would be impossible.

Addition Principle of Counting

There's also an addition principle of counting. This arises when we need to include objects from two different sets. For instance, we might need to calculate the probability that the people in a group are either from California or Texas. In this case, the favorable outcomes would include the union of those two sets, which requires the sets to be added. Here's the rule stated formally.

Table 80.3

Addition Principle of Counting	**n(A or B)=n(A)+n(B)**

Of course if the sets intersected, then we have to subtract the number in the intersection set. That might be the case if some people had homes in both California and Texas.

Table 80.4

Addition Principle of Counting with Intersection	**n(A or B)=n(A)+n(B)-n(A∩B)**

This is the same principle we used to calculate the probability of drawing a diamond or a face card from a deck. Only instead of drawing a Venn diagram, we could have just used the above formula.

Practice 80

a. Convert the repeating decimal $4.4444...$ into a fraction.

b. Use mathematical induction to prove that the equation $1+4+7+...+(3n-2)=\frac{n(3n-1)}{2}$ is true for all positive integer values of *n*.

c. There are 8 black, 4 navy, 2 blue and 6 white pairs of socks in a drawer. Without looking, Jim pulled out one pair. What is the probability that he chose a black or navy pair of socks?

d. In a group of 25 students, 8 students can play the piano, 13 students can play the guitar, and 4 students can play both piano and guitar. If one student is chosen at random from this group to participate in a musical event, what is the probability that this student can play the piano or the guitar?

e. A home security company offers a security system that uses a security code. How many different security codes can be created if each code has 4 letters followed by 3 numbers?

Problem Set 80

Tell whether each sentence below is True or False.

1. When an event is absolutely certain to happen, the probability is 100% (or 1), and when an event will definitely not happen, the probability is 0% (or 0).

2. To calculate probability, take the number of favorable outcomes divided by the total number of outcomes.

Answer each question below.

3. Find the 29$^{\text{th}}$ term in the arithmetic sequence with $a_1=8$ and a common difference (*d*) of $-\frac{1}{4}$.

4. If the common ratio (*r*) of a geometric sequence is 3 and $g_7=1,458$, find the first term of the sequence.

5. Find the sum of the whole numbers from 1 to 400.

Find the sum of each infinite geometric series, or indicate that the sum does not exist.

6. $1+5+25+125+\ ...$

7. $5-\frac{5}{4}+\frac{5}{16}-\frac{5}{64}+\ ...$

Convert each repeating decimal below into a fraction.

8. 0.6666...

(a) 9. 3.3333...

Use mathematical induction to prove that the equation below is true for all positive integer values of n.

(b) 10. $1+5+9+...+(4n-3)=n(2n-1)$

Answer each question below.

11. A box contains 5 red, 2 blue, and 3 white marbles. One marble is chosen at random.

What is the probability that the chosen marble is white? What is the probability that the chosen marble is blue?

12. If a die is rolled two times, what is the probability of rolling 2 fours? Write your answer as a fraction.

13. What is the probability of rolling a 9 with two dice? Write your answer as a fraction.

(c) 14. There are 3 bottles of grape juice, 6 bottles of orange juice, 9 bottles of apple juice, and 12 bottles of fruit punch in the cooler. Without looking, Matthew chose a bottle for himself. What is the probability that he chose a bottle of orange juice or apple juice?

(d) 15. In a group of 20 students, 5 students play on a basketball team, 9 students play on a soccer team, and 3 students play on both a basketball and soccer team. If one student is chosen at random from this group, what is the probability that he or she is on a basketball or soccer team? Write your answer as a fraction.

(e) 16. How many different automobile license plates can be created if each plate has 3 letters followed by 3 numbers?

Select the answer for each question below.

17. Which of the following is a y-intercept of the hyperbola $\frac{y^2}{25}-\frac{x^2}{64}=1$?

A. $(0,25)$ B. $(5,0)$ C. $(0,8)$
D. $(0,-5)$ E. $(0,625)$

18. If the pattern of the terms $2\sqrt{2}$, 8, $16\sqrt{2}$ … continues, which of the following would be the 8$^{\text{th}}$ term of the sequence?

A. 2^8 B. $(\sqrt{2})^8$ C. 2^9
D. $(2\sqrt{2})^7$ E. $(2\sqrt{2})^8$

19. The graph of which of the following is symmetric with respect to the origin?

A. $y = -|x+5|$ B. $y = x^2 - 4$ C. $y = x^3 - 4x$

D. $y^2 = x - 3$ E. $y = (x+1)^2$

Answer the question below. Estimate any irrational answers to two decimal places.

20. If $x^5 = 7^4$, then $x = ?$

21. What is the perimeter of a rectangle that has vertices $(1, \sqrt{3})$, $(1, 4)$, and $(4, 4)$?

Lesson 81—Permutations and Combinations

In the last lesson, we started learning about probability. Since counting all the possible outcomes (and drawing tree diagrams!) can be long and tedious for tough problems, we showed you some shortcuts for doing that.

Permutations

In one of our examples, we used the multiplication principle of counting to figure out how many possible passwords with 3 letters followed by 3 numbers could be written. It was possible to have passwords like AAA 333 and BBB 999, where the letters are numbers appear repeatedly. But in some problems, it doesn't make sense for the same object to appear more than once. Here's an example like that.

> Tom, Sue, and Bill are going on a plane trip, and their seat assignments have them all on the same row. How many different seating arrangements or permutations are possible?

We have 3 objects: Tom, Sue, and Bill. But obviously, seating arrangements like Tom-Tom- Tom or Sue-Sue-Sue aren't possible. A person can only be in one seat at a time! So we can only use an object once. This kind of problem is called a permutation. A **permutation** is just an arrangement of some or all of the objects (elements) from a given set in a definite order. Since it's an arrangement of specific objects, there's no way to use an object repeatedly. The first step for this problem is to draw blanks to represent the 3 seats on the row.

3 letters

— — —

There are 3 possibilities for the first seat, because nobody has sat down yet. But then there are only 2 possibilities for the second seat (since one person is already seated) and there's just 1 possibility for the last seat (since two people are already seated). We can think of each person sitting down as an event. Then using the multiplication principle of counting, we get the following.

$$3\times2\times1=6$$

There are 6 possible seating permutations (arrangements) for Tom, Sue, and Bill.

If 7 people were sitting down in a row with 7 seats, then the number of possibilities would be $7\times6\times5\times4\times3\times2\times1$ or 5,040 possible permutations. If 10 people were sitting down in a row with 10 seats, the number of possibilities would be $10\times9\times8\times7\times6\times5\times4\times3\times2\times1$ or 3,628,800 possible permutations! Notice the pattern is always the number of people/rows multiplied by each whole number all the way down to 1. This kind of calculation is used so much (especially with probabilities) that there's a special symbol for it. $3\times2\times1$ is written as $3!$ and pronounced "3 factorial." Similarly, the expression $7\times6\times5\times4\times3\times2\times1$ is written as $7!$ ("7 factorial"), and $10\times9\times8\times7\times6\times5\times4\times3\times2\times1$ is written as $10!$ ("10 factorial"). More generally, for n objects, where each can be used only once, the number of possible permutations is always $n!$ ("n factorial").

Table 81.1

n objects, where each can be used just once, have n! possible permutations (arrangements).

Sometimes there are more objects than spaces. For instance, what if 7 people are rushing to sit down in just 4 seats? How many possible permutations (arrangements) are there? It's not $7!$ because there aren't 7 seats available.

only 4 seats

— — — —

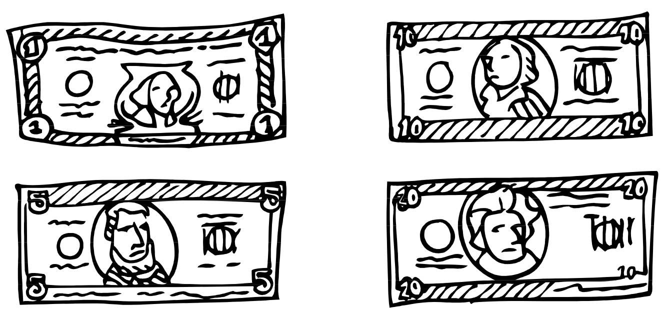
There are 7 possibilities for the first seat, since no one has sat down. There are only 6 possibilities for the second seat, 5 for the third seat, and 4 for the fourth seat. So the total number of permutations (arrangements) is $7\times6\times5\times4$ or 840. Generally, if there are *n* objects taken *r* at a time, then the number of permutations can be calculated as follows.

Table 81.2

Number of permutations of n objects taken r at a time.	$_nP_r=n(n-1)(n-2)\ldots(n-r+1)$

Using this formula to calculate the number of permutations for the 7 people rushing to fill 4 seats, we have $n=7$ and $r=4$. That gives us $7(7-1)(7-2)\ldots$ or $7\times6\times5$. And the pattern would continue until $n-r+1$ or $7-4+1$, which equals 4. So the calculation is $7\times6\times5\times4$ or 840, which we know is right. One short way to write this permutation is $_7P_4$. The little number on the left stands for *n* and the one on the right stands for *r*. So $_7P_4=840$. Generally, any permutation of *n* objects taken *r* at a time can be written as $_nP_r$.

There's one tricky situation involving permutations that we should discuss. What if we're counting the possible arrangements for a group of objects, but some of the objects are identical? One simple example involves arranging money. Let's say we have two \$20 bills, one \$10 bill, one \$5 bill and one \$1 bill.



How many different arrangements of these bills would be possible? There are 5 bills in total. Here are 5 blanks to represent the places we could put each of the bills.

spaces for 5 bills

— — — — —

The usual way to calculate this would be $5!$ or $5\times4\times3\times2\times1$, which equals 120. But there's one problem. Since we have two \$20 bills, some of these permutations would be the same. For instance, there are two permutations where the two \$20s are in the first two spaces and then the other spaces are \$10, \$5, and \$1.

These two permutations are really the same.

\$20 \$20 \$10 \$5 \$1   \$20 \$20 \$10 \$5 \$1

We say that these two permutations are not distinguishable, which means that we can't tell them apart. So we've double counted these two. In fact, we've double counted all the permutations, because the two \$20s can always be switched, and there won't be able to distinguish the new arrangement from the original. To adjust for this, we need to take the total number of permutations we calculated, which was 120, and divide it by 2! Since 2! Is the same as 2×1 or 2, that eliminates the double counting perfectly. The real number of permutations for this situation is 60.

$$\frac{120}{2!}=60$$

What if there had been three \$20 bills, one \$10 bill, one \$5 bill, and one \$1 bill? Well, that's 6 bills, so we would start with $6!$ or $6\times 5\times 4\times 3\times 2\times 1$, which is 720. But to adjust for the double counting, we would in this case have to divide 720 by $3!$. That's because for any given permutation, such as the one below, there would be $3\times 2\times 1$ or 6 permutations included in our 720.

These 6 permutations are all the same

\$20 \$20 \$20 \$10 \$5 \$1   \$20 \$20 \$20 \$10 \$5 \$1   \$20 \$20 \$20 \$10 \$5 \$1

\$20 \$20 \$20 \$10 \$5 \$1   \$20 \$20 \$20 \$10 \$5 \$1   \$20 \$20 \$20 \$10 \$5 \$1

We have 6 of these in the original 720, because there are 6 ways to arrange 3 objects. So to adjust for this, we need to divide 720 by $3!$ The real number of permutations, then, turns out to be $\frac{720}{3!}$ or 120. Generally, the number of permutations for n objects will be $n!$ divided by the factorial for the number of objects that are the same. If there are three \$20 bills, then we divide by $3!$. If there were three \$20 bills and two \$10 bills, we would need to divide by $3!2!$, which is the same as dividing by 12.

Combinations

In certain situations, the order in which objects are arranged doesn't make any difference. For example, what if 7 people (Steve, Pam, John, Cindy, Joe, Bruce, and Tara) are interviewing for 4 summer lifeguard jobs? How many possible groups can be hired? The order of the 4 people hired really makes no difference. If Steve, Cindy, Bruce, and Tara are hired, that's the same as Cindy, Bruce, Tara, and Steve. Since order doesn't matter we don't want to calculate the number of possible permutations. We need to figure out the number of combinations. A **combination** is an unordered arrangement of *n* objects (elements) taken *r* at a time.

To figure out the number of possible groups of lifeguards that could be hired, we start with the number of permutations, which is $7\times 6\times 5\times 4$ and then we eliminate the double counting. There's double counting, because the permutations assume Steve, Cindy, Joe, Bruce, and Tara is different from Cindy, Bruce, Tara, and Steve. The double counting is eliminated by dividing by 4!. Making this adjustment gives us 35 possible groups of lifeguards.

$$\frac{7\times 6\times 5\times 4}{4!}=35$$

Notice the number of combinations for 7 objects taken 4 at a time is a lot smaller than the number of permutations. Not paying attention to the order makes a big difference. We use the letter *C* for combinations, so the number of combinations for 7 objects taken 4 at a time is written as ${}_7C_4$. Here's the general rule for calculating combinations.

Table 81.3

Number of combinations (without regard to order)
of n objects taken r at a time: ${}_nC_r=\frac{n(n-1)(n-2)...(n-r+1)}{r!}$

Different Ways of Writing

The rules for calculating permutations and combinations are written several different ways. Instead of writing a permutation of n objects taken r at a time as ${}_nP_r = n(n-1)(n-2)...(n-r+1)$, it's also written as ${}_nP_r = \frac{n!}{(n-r)!}$. The two expressions are identical mathematically. It's just that $\frac{n!}{(n-r)!}$ is easier to write. To see why these are the same, think of a case of 9 objects taken 4 at a time. Using the formula $n(n-1)(n-2)...(n-r+1)$, we would have $9\times8\times7\times6$. The second formula, $\frac{n!}{(n-r)!}$, works out to $\frac{9!}{(9-4)!}$, which is the same as $\frac{9\times8\times7\times6\times5\times4\times3\times2\times1}{5\times4\times3\times2\times1}$. Canceling the $5\times4\times3\times2\times1$'s, we still end up with $9\times8\times7\times6$. It works the same way for any other permutation. So $n(n-1)(n-2)...(n-r+1)$ and $\frac{n!}{(n-r)!}$ are both acceptable formulas for a permutation. The formula for a permutation turns out to be a very important one in many different areas of math. It's so important that an even shorter symbol is often used. Instead of $n(n-1)(n-2)...(n-r+1)$ or $\frac{n!}{(n-r)!}$, the formula is also written as $\binom{n}{r}$.

Remember, a combination of n objects taken r at a time is just the formula for a permutation divided by $r!$. But since $n(n-1)(n-2)...(n-r+1)$ and $\frac{n!}{(n-r)!}$ are the same, we can also write the formula for a combination as $\frac{\frac{n!}{(n-r)!}}{r!}$, which simplifies to $\frac{n!}{r!(n-r)!}$. Here are the different ways of writing the formulas for permutations and combinations.

Table 81.4

Permutation formulas	$n(n-1)(n-2)...(n-r+1)$
	$\frac{n!}{(n-r)!}$
	$\binom{n}{r}$
Combination formulas	$\frac{n(n-1)(n-2)...(n-r+1)}{r!}$
	$\frac{n!}{r!(n-r)!}$

Practice 81

a. Convert the repeating decimal $0.363636...$ into a fraction.

b. How many different ways can the letters of the word SESSION be arranged?

c. A cube has edges of length 4. If M and N are points on its surface, what is the maximum straight-line distance from M to N?

A. $4\sqrt{2}$ B. 4 C. $8\sqrt{2}$
D. 12 E. $4\sqrt{3}$

d. In an arithmetic sequence, $a_8 = 9$ and $a_{12} = 37$. What is a_{103}?

e. In the geometric series where $g_1 = 5$ and $g_{n+1} = 6g_n$, what is $g_6 - g_2$?

f. Solve the word problem below.

While scaling down a picture on the computer, Kyle noticed that the size of the picture he was scaling grew smaller on the screen. If the width and height (in centimeters) can be represented by the parametric equations $w(t) = 8(.99)^t$ and $h(t) = 12(.98)^t$ where t is time in seconds. Based on this information, what would the area of the picture be on the screen after 3 seconds? Estimate your answer to two decimal places.

Problem Set 81

Tell whether each sentence below is True or False.

1. A permutation is an arrangement of some or all of the objects (elements) from a given set in a definite order.

2. A combination is an unordered arrangement of n objects (elements) taken r at a time.

Tell whether each conic section below is a circle, parabola, ellipse or hyperbola without completing the square?

3. $8x^2 + 8y^2 + 10x - 26y + 42 = 0$

4. $4x^2 - 26x + 5y - 186 = 0$

Find the sum of each infinite geometric series, or indicate that the sum does not exist.

5. $6 + \frac{9}{2} + \frac{27}{8} + \frac{81}{32} + \ ...$

6. $\frac{1}{4} + \frac{1}{2} + 1 + 2 + \ ...$

Convert each repeating decimal below into a fraction.

7. 0.1111...

(a) 8. 0.626262...

Answer each question below.

9. There are 7 blue, 2 white, and 5 black notebooks on a shelf. What is the probability of selecting from the shelf a notebook that is white without looking? Write your answer as a fraction.

10. What is the probability of rolling a 7 with two dice? Write your answer as a fraction.

11. What is the probability of drawing a king or queen from a deck of 52 cards? (Hint: There's a king and queen for each of the 4 kinds of cards in a deck.) Write your answer as a fraction.

12. In a linguistics class having 30 students, 12 students can speak English, 9 students can speak Spanish, and 3 students can speak both English and Spanish. If one student is chosen at random from this class, what is the probability that this student can speak English or Spanish?

Calculate the number of permutations in each problem below.

13. How many possible ways can a group of 6 students line up to buy tickets to a play?

14. If 9 people are rushing to sit down in just 5 seats, how many possible permutations (arrangements) are there?

(b) 15. How many different ways can the letters of the word PIZZA be arranged?

Calculate the number of combinations in each problem below.

16. How many different 6-member student committees can be formed from a group of 12 candidates?

17. Sara has 7 magazines. If she wishes to bring 4 of them with her on vacation, how many different combinations of 4 magazines are possible?

Select the answer for each question below.

18. For $x \neq 0$, if $3^{-2} - 5^{-2} = x^{-2}$, then $x = ?$

A. $3\sqrt{3}$ B. $2\sqrt{3}$ C. $\pm\frac{5}{2}$

D. $\pm2\sqrt{3}$ E. $\pm\frac{15}{4}$

(c) 19. A cube has edges of length 3. If *P* and *Q* are points on its surface, what is the maximum straight-line distance from *P* to *Q*?

A. $3\sqrt{2}$ B. 9 C. $3\sqrt{3}$

D. $2\sqrt{3}$ E. $6\sqrt{2}$

Answer each question below.

(d) 20. In an arithmetic sequence, $a_7 = 8$ and $a_{10} = 23$. What is a_{100}?

(e) 21. In the geometric series where $g_1 = 3$ and $g_{n+1} = 4g_n$, what is $g_5 - g_2$?

Solve the word problem below.

(f) 22. While zooming slowly in on a GPS satellite map, Eleanor noticed that the size of the building she was looking for grew larger on the screen. If the width and height (in centimeters) can be represented by the parametric equations $w(t) = 3(1.1)^t$ and $h(t) = 2(1.1)^t$ where t is time in seconds. Based on this information, what would the area of the building be on the screen after 3 seconds? Estimate your answer to two decimal places.

Lesson 82—More on Probability

We've been studying probability, and learning methods for counting up all the possible outcomes and favorable outcomes. Fortunately, it's not always necessary to do so much counting. There are probability shortcuts that can be used. Here's an example.

> At the birthday party, Sarah was blindfolded and asked to pull 2 balls out of a bag containing a total of 12 balls. If 8 of the balls in the bag are red and 4 are green, what is the probability that both the balls Sarah pulls out will be green?

We could rely on the combination formula on this problem to figure out the total possible outcomes. There are 12 objects taken 2 at a time. Since order doesn't matter, we should use combinations instead of permutations. The formula is ${}_nC_r = \dfrac{n(n-1)(n-2)...(n-r+1)}{r!}$, as you know. That gives us a total of 66 possible outcomes.

$$ {}_{12}C_2 = \frac{12 \cdot 11}{2!} = 66 $$

The favorable outcomes are all the possibilities where both of the balls pulled out are green. Since order doesn't matter again, these are also combinations. We're taking 4 green balls 2 at a time, so that's ${}_4C_2$. The number of favorable outcomes turns out to equal 6.

$$ {}_4C_2 = \frac{4 \cdot 3}{2!} = 6 $$

Now we have both the possible outcomes and the favorable outcomes. The last step is to figure out the probability by dividing.

Pulling out 2 green balls

$$ \frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes}} = \frac{6}{66} = \frac{1}{11} $$

The probability of pulling out 2 green balls is $\frac{1}{11}$ or about 9.1%.

Probability Shortcuts

Instead of using combinations, another (shorter) method for doing the problem is to calculate the probability of pulling each green ball individually. This is shorter, because the number of possible outcomes is smaller for each individual event. There are 12 balls in total and 4 of them are green, so the probability of pulling out one green ball is $\frac{4}{12}$ or $\frac{1}{3}$.

Pulling out 1st green ball

$$ \frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes}} = \frac{4}{12} = \frac{1}{3} $$

With one green ball already pulled out, there are 11 balls left in the bag and 3 of them are green. So the probability of pulling out a green ball the second time is $\frac{3}{11}$.

Pulling out 2nd green ball

$$\frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes}} = \frac{3}{11}$$

But how do we get the final probability of pulling out two green balls from $\frac{1}{3}$ and $\frac{3}{11}$? It's pretty easy, actually. We can just multiply the two individual probabilities together.

Multiply the 1st and 2nd probabilities

$$\frac{1}{3} \cdot \frac{3}{11} = \frac{1}{11}$$

This is called the multiplication principle of probability. It basically says that the probability of some event *A* multiplied by the probability of some other event *B* is equal to the probability that both *A* and *B* occur. That's true as long as the probability of *B* assumes that *A* did actually occur. When calculating the probability of pulling out the second green ball, we assumed that one green ball had already been removed. If two events have no affect on each other, then there's no need to worry about adjusting the second probability. Events that don't affect each other are called independent events. Here's the multiplication principle of probability stated formally.

Table 82.1

Multiplication Principle of Probability	**If event A has a probability of p_1 and event B has a probability of p_2 (assuming A occurs), then the probability that both A and B occur is p_1p_2**

There's also a shortcut rule for calculating the probability that either event A *or* event B will occur. Since the occurrence of either event is a favorable outcome, we have to add the two probabilities together. For example, what if we wanted to calculate the probability of a coin landing heads or tails in a flip? The probability of heads is $\frac{1}{2}$ and the probability of tails is $\frac{1}{2}$. To get the probability of either one, we add those two probabilities together.

probability of a coin landing heads or tails

$$\frac{1}{2} + \frac{1}{2} = 1$$

The result is 1 or 100%, which makes perfect sense. The coin has to land on one side or the other, so there must be a 100% chance that it will end up either heads or tails. This shortcut is called the Addition Principle of Probability, and it's stated formally below.

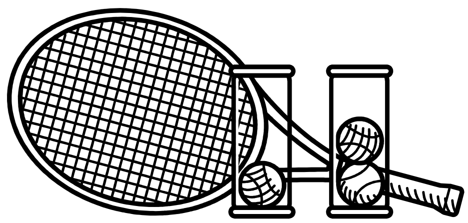
Table 82.2

Addition Principle of Probability	**If event A has a probability of p_1 and event B has a probability of p_2 (assuming A occurs), then the probability that either A or B occur is p_1+p_2**

One important point about the above Addition Principle is that it requires the two events to be "mutually exclusive." That just means if one event happens, then the other event won't happen, and vice versa. When flipping a coin, heads and tails are clearly mutually exclusive. (A coin can't land both heads and tails.) The different possibilities when rolling a die are also mutually exclusive. But what about picking a piece of chocolate or crème-filled Valentine's candy out of a box? Those aren't mutually exclusive, because it's possible for a piece of candy to be both chocolate and crème-filled at the same time. So we couldn't add the probabilities for chocolate and crème-filled together to get the probability of picking either one.[4]

Unequally Likely Outcomes

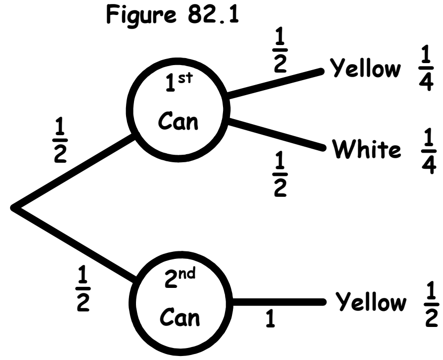
It to now, all of our probability problems have involved equally likely outcomes. For example, when flipping a coin, heads are as likely as tails. And when rolling a die, a 1 is just as likely as a 2, 3, 4, 5, or 6. But in some probability problems the outcomes are not equally likely. Take this example.



Two identical tennis ball cans are sitting next to each other. The first contains 1 yellow tennis ball and 1 white tennis ball. The second contains 1 yellow tennis ball. What's the probability of choosing a yellow tennis ball (without looking, of course)?

At first it might seem like we should just add up all the tennis balls (3) and divide by the number of yellow ones (2) to get an answer of $\frac{2}{3}$. And $\frac{2}{3}$ would be the correct answer, if all the balls were in the same can. But since they're in different cans, the 3 balls are not equally likely outcomes. Since it's by itself, the single yellow ball in the second can has a better chance of being chosen than either of the 2 balls in the first can. We have to take that into account in our calculation. Selecting a ball is a two step process. The first step is to choose a can. Since there are two cans (and both are equally likely), the probability of picking either can is $\frac{1}{2}$ or 50%. The first can contains a yellow ball and a white ball. So if the first can is chosen, there is a $\frac{1}{2}$ probability of choosing the yellow ball and a $\frac{1}{2}$ probability of choosing the white ball. If the second can is chosen, since that can only has one yellow ball, there is a probability of 1 (or 100%) of choosing it. The tree diagram below shows the analysis.

Figure 82.1



[4] In such cases, what we have to do is calculate the probability that a piece of candy has both chocolate and crème, then subtract that probability from the answer.

To figure out the probability of choosing the top yellow ball (from the first can), we can use the multiplication principle of probability. It takes two events to get to that ball. We have to choose the first can and then we have to choose the yellow ball (instead of the white one) in that can. Multiplying those two probabilities gives us $\frac{1}{2} \cdot \frac{1}{2}$ or $\frac{1}{4}$. So the probability of choosing the yellow ball from the first can is $\frac{1}{4}$. The probability is the same for choosing the white ball in the first can. There's a $\frac{1}{2}$ probability of choosing the first can and a $\frac{1}{2}$ probability of choosing the white ball: $\frac{1}{2} \cdot \frac{1}{2}$ equals $\frac{1}{4}$. The probability of choosing the yellow ball in the second can is different. We have to choose the second can. That's a $\frac{1}{2}$ probability. But once that can is chosen, there's a probability of 1 of choosing the yellow ball, since it's the only ball in the can. So the probability of choosing this yellow ball is $\frac{1}{2} \cdot 1$ or $\frac{1}{2}$. Notice that the probabilities of choosing the 3 balls are different. The bottom yellow ball (in the second can) is more likely than the other two. That's why we couldn't just take 2 (the favorable outcomes) and divide it by 3 (the total outcomes) to get our answer.

But how do we finish the problem and calculate the probability of choosing any yellow ball? Well, we know the probability of choosing each yellow ball individually. Since those are both favorable outcomes, we can add them together. According to the Addition Principle of Probability, this will give us the probability of choosing either yellow ball.

$$\frac{1}{4} + \frac{1}{2} \text{ equals } \frac{3}{4} \text{ or } 75\%$$

The probability of choosing a yellow ball in this situation is $\frac{3}{4}$. Importantly, the answer is different from $\frac{2}{3}$, which is the probability when all of the balls are mistakenly treated as equally likely. Another interesting thing is that the probabilities of all the possible outcomes still add to equal 1. The probabilities are $\frac{1}{4}$ for the top yellow ball, $\frac{1}{4}$ for the white ball, and $\frac{1}{2}$ for the bottom yellow ball. Adding those gives us $\frac{1}{4} + \frac{1}{4} + \frac{1}{2}$ or 1. This is no coincidence. No matter what the probabilities of the individual outcomes, the probabilities of all the outcomes added together must add to equal 1.

Expected Value

Probability calculations are very useful in the real world, particularly when money is involved. For instance, let's say you're about to mail a package containing a \$100 item. Then the clerk at the post office asks whether you would like to buy insurance. The insurance will pay you \$100 if the post office loses the package. Let's say that the probability of losing the package is 5% and the insurance charge is \$10. Is it wise financially to buy the insurance?

There are two possible outcomes: the package is lost or not lost. Those outcomes are not equally likely, because we said the probability of losing the package is just 5% or 0.05. Since the probability of all the outcomes must add to equal 1, the probability of not losing the package must be $1 - 0.05 = 0.95$ or 95%. So these are pretty lopsided probabilities. The way to figure out whether the insurance makes sense is to use the concept of expected value. The **expected value** of an event is the amount of money you will receive or pay if the event occurs multiplied by the probability of the event happening. The probability of the post office losing the package is 5%. Assuming you purchase the insurance, the post office will pay you \$100 of that happens. So the expected value of losing the package is 100 times 5% or \$5. That's the value of the insurance. Yet the post office is charging \$10, which is twice the expected value. You should be willing to pay only up to \$5 for the insurance.

Here's another example. What if you are thinking of playing the lottery to try and win millions of dollars? It only costs \$1 to play and the lottery winner this week will receive \$6,000,000. The way this lottery works is that you choose 6 numbers, each one between 1 and 55. If your number is chosen, you get the \$6,000,000! First, let's figure out the probability of winning. There are 55 choices for each of the 6 numbers.

Choosing 6 different numbers, each between 1 and 55

— — — — — —

Since the same number cannot be repeated, we need to figure out the permutations. The total possible outcomes are equal to $55 \cdot 54 \cdot 53 \cdot 52 \cdot 51 \cdot 50$ or 20,872,566,000. So nearly 21 billion numbers can be chosen. Assuming they're all equally likely, the probability that your number will be the winner is 1 (the favorable outcome) divided by 20,872,566 (the possible outcomes).

$$\frac{1}{20{,}872{,}566{,}000}$$ **probability of winning**

For the lottery to be a good deal, the expected value should be at least \$1, since that's how much it costs to play. The expected value is $\frac{1}{20{,}872{,}566{,}000}$ times \$6,000,000.

$$\left(\frac{1}{20{,}872{,}566{,}000}\right)(\$6{,}000{,}000) \approx \$0.0029$$ **expected value**

The expected value is about \$.0029 dollars or 0.29 cents. So at a price of \$1, this lottery is charging over 300 times the expected value. That's a bad deal!

Practice 82

a. Write a direct formula for the arithmetic sequence 9, 6.5, 4, 1.5, …?

A. $9-2.5^{n-1}$ B. $11.5-2.5n$ C. $2.5n+9$
D. 2.5^n E. $2.5n$

b. In how many ways can the letters of the word INITIALS be arranged using all of the letters?

c. A bag contains 7 pennies, 4 nickels and 5 dimes. Without looking, two coins are drawn out of the bag. What is the probability of selecting 2 dimes? Write your answer as a fraction.

d. At a local neighborhood association, a new board member is chosen by picking a name without looking from two candidate boxes. In the first box, there are names of 5 men and 3 women. In the second box, there are names of 2 men and 6 women. What is the probability of choosing a woman as the new board member? Write your answer as a fraction.

e. If $x_1 = 4$ and $x_{n+1} = \sqrt[3]{-16x_n}$, then $x_5 = ?$

Problem Set 82

Tell whether each sentence below is True or False.

1. The multiplication principle of probability says that the probability of some event *A* multiplied by the probability of some other event *B* is equal to the probability that both *A* and *B* will occur.

2. The expected value of an event is the amount of money that you have to pay if the event doesn't occur.

Select the correct answer for each sequence below.

(a) 3. Write a direct formula for the arithmetic sequence 8.5, 8, 7.5, 7, …?

A. $0.5n+8.5$ B. 0.5^n C. $8.5-0.5^{n-1}$
D. $9-0.5n$ E. $0.5n$

4. Write a recursive formula (n^{th} term) for the geometric sequence $2,\ 8,\ 32,\ 128\ldots$?

A. $a_n = a_{n-1}+6$ B. $a_n = 4a_{n-1}$ C. $a_n = 4^n$

D. $a_n = 4n$ E. $a_n = \frac{1}{4}a_{n-1}$

Find the sum of each infinite geometric series, or indicate that the sum does not exist.

5. $18+6+2+\frac{2}{3}+\ ...$

6. $30-6+\frac{6}{5}-\frac{6}{25}+\ ...$

Calculate the number of permutations in each problem below.

7. How many possible ways can 5 paintings be displayed on a wall?

8. If 10 people are rushing to sit down in just 6 seats on a subway, how many possible permutations (arrangements) are there?

(b) 9. In how many ways can the letters of the word SEQUENCE be arranged using all of the letters?

Calculate the number of combinations in each problem below.

10. How many different 8 member teams can be chosen from a group of 17 student athletes?

11. A teacher has a test bank of 14 questions. If she wishes to create a test using 9 of the questions, how many different combinations are possible?

Answer each question below.

12. Find the constant difference (d) for the arithmetic sequence with $a_1 = 16$ and $a_{23} = 104$.

13. Many computer programs come with serial numbers which are used against theft. How many different serial numbers can be created if each number has 4 letters followed by 2 numbers?

(c) 14. A bag contains 5 pennies, 3 nickels and 4 dimes. Without looking, two coins are drawn out of the bag. What is the probability of selecting 2 nickels? Write your answer as a fraction.

15. When rolling a die, what is the probability of rolling a 3 or a 6? Write your answer as a fraction.

(d) 16. At a local neighborhood association, a new board member is chosen by picking a name without looking from two candidate boxes. In the first box, there are names of 3 men and 2 women. In the second box, there are names of 1 man and 4 women. What is the probability of choosing a man as the new board member?

Select the answer for each question below.

17. If $f(x) = x^2 + 8$ for all $x \geq 0$, then the graph of $f^{-1}(x)$ intersects the x-axis at ?

A. f^{-1} is undefined
B. Exactly three points
C. Exactly two points
D. Exactly one point
E. Zero points

18. If A is a 2×5 matrix, and B is a 5×3 matrix, then the product of A times $2B$ is a matrix of which of the following dimensions?

A. 10×3
B. 2×6
C. 2×5
D. 5×5
E. 2×3

19. The product of the roots of a quadratic equation is -3 and their sum is -2. Which of the following could be the quadratic equation?

A. $x^2 + 3x - 2 = 0$
B. $x^2 + 2x - 3 = 0$
C. $x^2 - 3x + 2 = 0$
D. $x^2 - 2x + 3 = 0$
E. $x^2 - 2x - 3 = 0$

Answer each question below.

20. In a given geometric sequence, $g_3 = 5$ and $g_6 = 40$. What is $g_{12} \div g_{16}$?

(e) 21. If $x_1 = 3$ and $x_{n+1} = \sqrt[3]{-9x_n}$, then $x_4 = ?$

Lesson 83—Statistics

Statistics is the study of large groups of numbers. Corporations, scientists, medical researchers, economists, investors, and many others use statistics extensively in their work. It's one of the most practical and useful branches of mathematics. What makes statistics so important is that it allows us to make predictions about very complicated situations.

Experimental Probability

One example is insurance. An automobile insurance company agrees to pay you money if you wreck your car. You can then use the money to get the car fixed. The insurance company is willing to do this, if you agree to pay them a fee each month, whether you wreck your car or not. In order for the insurance company to make a profit, it has to collect more from its customers' monthly payments, than it pays when some of those customers wreck their cars.

Obviously, it's important for the insurance company to know the probability that you will wreck your car. Otherwise, they wouldn't know how much to charge in monthly payments. But how could the company determine such a probability? There's no way to find out the total possible outcomes for an individual driver. They don't know where you're going to drive, at what speed, or what the weather conditions are going to be. They don't know anything about other drivers you will meet on the road, or about the mechanical condition of your car. This is obviously way too complicated a situation to do a probability calculation. It's nothing like flipping a coin or rolling dice.

The insurance company can still figure out how much to charge, though, by using statistics. Instead of trying to analyze the situation for an individual driver, they look at driving records for millions of drivers. If 10,000,000 drivers had 430,000 wrecks last year, then they make 10,000,000 the possible outcomes and 430,000 the "favorable" outcomes and calculate the probability like this.

$$\frac{\text{number of wrecks}}{\text{number of drivers}} = \frac{430{,}000}{10{,}000{,}000} = 0.043$$

They then assume that the probability of a driver having a wreck is 0.043 or 4.3%. This turns out to be pretty accurate, because it's based on many, many drivers. So even though the insurance company doesn't know that much about whether any one person will wreck his car, they can be pretty sure that, out of all the people they insure, roughly 4.3% will have a wreck. The company's monthly charges are based on this number. In practice, the insurance company makes the probability calculation even more accurate by considering the age of the drivers, what kind of cars they drive, their past driving record, and so on. But the main point is that a fairly accurate measure of probability in a very complicated situation can be obtained by analyzing a large group of numbers. This is called experimental probability, because it's based on experience—the driving records of millions of people—rather than an analysis of an individual case.[5] That's just one example of how statistics is used in the real world.

[5] Figuring the probability of flipping heads or rolling a 6 on a die is called "theoretical probability."

Mean, Median, and Mode

A large group of numbers is called data. So statistics is all about analyzing data. And there are quite a few methods that are used. The simplest involve calculating the center point of a group of data. As an example, the table below shows the annual incomes in thousands of dollars for 15 families in a particular neighborhood.

Table 83.1

Family	Annual Income (in thousands)	Family	Annual Income (in thousands)	Family	Annual Income (in thousands)
Johnson	42	Martin	22	Montagne	43
Smith	38	Tafuri	31	Li	45
Henderson	45	Benson	50	Kluge	29
Brock	27	Tully	44	Billington	38
Gonzalez	44	Greer	61	Darnell	44

One way to measure the center of this data is to calculate the average of all the numbers. The technical name for average is mean. All we have to do is add up all 15 numbers and divide by 15. The sum is 603, and dividing that by 15 gives us a mean of 40.2. So a measure of the "center" of this data is 40.2 or $40,200. Generally, the mean of a group of numbers can be calculated by adding up all the numbers and dividing by however many numbers are in the group. Here is the formal definition.

Table 83.2

Definition of Mean	The mean of a list of n numbers $\{x_1, x_2 \ldots x_n\}$ is $\overline{x} = \dfrac{x_1 + x_2 + \ldots x_n}{n}$

An *x* with a line over it is the symbol for mean. The fraction on the right shows *n* numbers added together. And that sum is divided by *n*. We can also write the mean formula using sigma notation as $\frac{1}{n}\sum_{i=1}^{n} x_i$. This says to add x_1 through x_n (since *i* starts with 1 and ends with *n* on top), and multiply the total by $\frac{1}{n}$, which is the same as dividing by *n*.

But now let's say that Mr. Greer of the Greer family is a big movie star. So instead of an annual income of 61,000, the Greer's income is $12,000,000. Since the data is in thousands, it would go in the table as 12,000. How would this change the mean? The new total of all 15 incomes would be 12,542. Dividing by 15 gives a mean of 836.1 or $836,100 in annual income. Is this a good measure of the "center" of the data? It's really not. None of the other families have anywhere near this much income. The Greer income of $12,000,000 is called an "outlier." It lies far outside of the other data. Outliers distort the mean calculation and reduce its usefulness.

When a set of data has outliers (which can be very high or very low, compared to the other data), a better measure of the "center" of the data is something called the median. The **median** is the number that's exactly in the middle of the data. Let's order all of the incomes highest to lowest. And we'll continue to assume that Mr. Greer is a movie star so that his family income is $12 million.

Table 83.3

Family	Annual Income (in thousands)	Family	Annual Income (in thousands)
1. Greer	12,000	9. Johnson	42
2. Benson	50	10. Smith	38
3. Henderson	45	11. Billington	38
4. Li	45	12. Tafuri	31
5. Gonzalez	44	13. Kluge	29
6. Tully	44	14. Brock	27
7. Darnell	44	15. Martin	22
8. Montagne	43		

Since there are 15 numbers, the median is the 8^{th} number down. That's right in the middle, because there are 7 numbers higher and 7 numbers lower. The 8^{th} number is 43, which is the Montagne's income. Obviously, $43,000 is a much better measure of the center of the data than the mean of $836,100, because of the Greer's outlier income. We could have found the median just as easily by listing the incomes from lowest to highest. It doesn't make any difference which way you do it.

There are 15 incomes in our data. Since 15 is an odd number, the number right in the middle is the median. But what if there are 16 numbers or 18 or any other even number? Then there's no number right in the middle. In that case, the median is the mean of the two numbers in the middle of data. For instance, with 16 numbers in a group of data, the two middle numbers would be the 8^{th} and 9^{th} on the list. If those were 44 and 45, we would just figure out the mean, which is $\frac{44+45}{2}$ or 44.5. Then we take 44.5 as the median. Here is the formal definition of median.

Table 83.4

Definition of Median	**The median of a list of n numbers $\{x_1, x_2 \ldots x_n\}$ arranged in order (either ascending or descending) is the middle number if n is odd. If n is even, the median is the mean of the two middle numbers.**

There's a third measure of the center of a group of data called the mode. The **mode** is the number that appears most often in the data. The mode of the family incomes is 44 (or $44,000). This number appears three times in the data—more than any other number. Under what circumstances is the mode a good measure to use? It's most useful when a number appears really often for some important reason. As an example, what if we were analyzing the age at which people obtain a driver's license? We could calculate the mean age or the median. But if we analyzed the data, we would find that the vast majority of people obtain their driver's license at age 16. In other words, 16 would be the mode of the data. That's because 16 is the minimum driving age (in the U.S.) and most people get their driver's license as early as possible. That's a more important fact than the mean or median age for obtaining a driver's license. Here is the formal definition of the mode.

Table 83.5

Definition of Mode	**The mode of a list of numbers is the number that appears most frequently on the list.**

Weighted Mean

In certain problems, a mean (or average) calculation needs to be weighted so that some of the numbers being added count for more than others. One simple example is calculating a grade average. When figuring out a student's grade, exams usually count for more than daily assignments. Here's a specific problem.

> The final exam in an English course counts for 30% of the total grade, the two semester tests count for 10% each, and all the daily assignments count for the remaining 50%. If Mark scored 90 on the final exam, 82 and 87 on the two semester tests, and had an average of 74 on all the daily assignments, what is his final grade in the course?

The problem gives us 4 of Mark's grades. To calculate his final grade, we can't just add up those 4 numbers and divide by 4 to get the mean. That's because some grades are worth more than others. What we have to do is weight each grade according to the percentage of the final grade that it's supposed to represent. To weight a number in a mean calculation, all we have to do is multiply the number by the appropriate percentage. Here is the calculation for Mark's final grade.

$$\bar{x} = 0.3(90) + 0.1(82) + 0.1(87) + 0.5(74)$$

Notice that we don't divide by anything when calculating a weighted mean. That's not necessary as long as the weight percentages all add to equal 100% (or 1 when written as decimals). To get the answer, we just simplify the right side.

$$\bar{x} = 27 + 8.2 + 8.7 + 37$$

$$\bar{x} = 80.9$$

The weighted mean comes out to 80.9 or 81 when rounded to the nearest whole number, So Mark got a low B for his English course.

Practice 83

a. Convert the repeating decimal $0.434343...$ into a fraction.

b. If two dice are rolled, what is the probability that the sum of the numbers on the dice comes out to 3 or 8? Write your answer as a fraction.

c. The probability that there will be a snow storm in New York tomorrow is $\frac{1}{4}$ and, independently, the probability that there will be a snow storm in Honolulu tomorrow is $\frac{1}{7}$. What is the probability that there will be a storm tomorrow in New York but not in Honolulu? Write your answer as a fraction.

d. The final exam in a Chemistry course counts for 30% of the total grade, the two semester tests count for 25% each, and all the homework assignments count for the remaining 20%. If Mark scored 75 on the final exam, 87 and 83 on the two semester tests, and had an average of 91 on all the daily assignments, what is his weighted mean grade in the course?

e. If $\frac{n!}{5} = (n-1)!$ then $n = ?$

f. Solve the word problem below.

Theodore's average fuel consumption on his first nine tanks was 31.8 miles per gallon. If he gets 27.5 miles per gallon on his next tank, what would his new average be? Round your answer to one decimal place.

Problem Set 83

Tell whether each sentence below is True or False.

1. The technical name for average is median.

2. The mean is the number that's exactly in the middle of a group of data.

Convert each repeating decimal below into a fraction.

3. 0.3333...

(a) 4. 0.151515...

Calculate the number of permutations in each problem below.

5. If 13 people are rushing to sit down in just 5 seats, how many possible permutations (arrangements) are there?

6. How many different ways can the letters of the word REPLACEMENT be arranged?

Calculate the number of combinations in each problem below.

7. How many 4-person committees can be formed from a group of 9?

8. Jane has 12 novels. If she wishes to bring 5 of them with her on her cruise, how many different combinations of 5 novels are possible?

Answer each question below.

9. In a factory, 3 tools are used to manufacture a certain product. There is a 0.09 probability that each tool will result in an unacceptable production error. If 3 tools are used, what is the probability that all of them will result in an unacceptable production error? Write your answer as a decimal estimated to 5 places.

10. A golf club consists of 12 members and is holding officer elections to select a president, secretary, public relations person and treasurer for the club. A member can only be selected for one position. How many possibilities are there for selecting the four officers?

(b) 11. If two dice are rolled, what is the probability that the sum of the numbers on the dice comes out to 4 or 7?

(c) 12. The probability that there will be a snow storm in Chicago tomorrow is $\frac{1}{6}$ and, independently, the probability that there will be a snow storm in Copenhagen tomorrow is $\frac{1}{9}$. What is the probability that there will be a storm tomorrow in Chicago but not in Copenhagen? Write your answer as a fraction.

13. There are 5 French, 3 Spanish and 10 English books on a shelf. Without looking, two books are drawn from the shelf. What is the probability of selecting 2 English books? Write your answer as a fraction.

Answer each question below.

The sales for the year 2006 (in thousand dollars) of 9 clothing stores in a shopping mall were as follows: 750, 1,000, 1,200, 1,400, 500, 1,500, 1,350 1,400, 800.

14. Find the mean of the clothing store sales.

15. Find the median and mode of the clothing store sales.

(d) 16. The final exam in a Physics course counts for 35% of the total grade, the two semester tests count for 20% each, and all the homework assignments count for the remaining 25%. If Mark scored 80 on the final exam, 95 and 70 on the two semester tests, and had an average of 86 on all the daily assignments, what is his weighted mean grade in the course?

Select the answer for each question below.

17. What is the domain of $f(x) = \dfrac{1}{x^2+5}$?

A. $-5 \le x \le 5$ B. $x \ne \pm\sqrt{5}$ C. All real numbers
D. $-\sqrt{5} \le x \le \sqrt{5}$ E. $x \ne \pm 5$

18. $\cot\alpha\cos\alpha + \sin\alpha = ?$

A. $\sec\alpha$ B. $2\sin\alpha$ C. 1
D. $\sin\alpha + \csc\alpha$ E. $\csc\alpha$

Answer each question below.

19. If $\log_a s = 5$ and $\log_a t = 10$, then $\log_a \dfrac{s}{t^2} = ?$

20. If 1, 4, and 7 are the first three terms of an arithmetic sequence, what is the sum of the first 16 terms of the sequence?

(e) 21. If $\dfrac{n!}{3} = (n-1)!$ then $n = ?$

Solve the word problem below.

(f) 22. Gary's average score on his first six tests is 87. If he earns a 90 on the seventh test, what will his new average be? Estimate your answer to one decimal place.

Lesson 84—More on Statistics

In the last lesson we learned how to calculate the mean, median, and mode of a group of data. Those are all different measures for the "center" of the data. Another important thing to measure about data, though, is its dispersion or spread.

Range and IQR

The spread is basically how far apart the numbers are. The simplest measure of spread is called the **range**. That's the difference between the highest and lowest number in the data. Let's look again at the family incomes that we worked with in the previous lesson.

Table 84.1

Family	Annual Income (in thousands)	Family	Annual Income (in thousands)
1. Greer	61	9. Johnson	42
2. Benson	50	10. Smith	38
3. Henderson	45	11. Billington	38
4. Li	45	12. Tafuri	31
5. Gonzalez	44	13. Kluge	29
6. Tully	44	14. Brock	27
7. Darnell	44	15. Martin	22
8. Montagne	43		

The highest income is 61, for the Greer family. And the lowest income is 22, for the Martin family. That means the range of the data is $61-22$ or 39. That's one way to measure the spread of this data. Of course, if we assume that Mr. Greer is a movie star, with an income of \$12 million, then the range would be $12{,}000-22$ or 11,978, which is much larger. Outliers can obviously have a big impact on the range. Here's a formal definition of range.

Table 84.2

Definition of range	**The range is the difference between the maximum and minimum values in a group of data.**

There's another measure of the spread, called the interquartile range, that's less easily distorted by outliers. The **interquartile range** (*IQR*) is the range of the middle half of the data. To calculate the *IQR*, we first find the median and split the data in half. Next, we find the quartiles. The first quartile (Q_1) is the median of the lower half of the data. The second quartile (Q_2) is the median itself (for the entire group of data). And the third quartile (Q_3) is the median of the upper half of the data. These three numbers separate the data into fourths.

$$\mathbf{61, 50, 45,}\quad \overset{Q_1}{\mathbf{45,}}\quad \mathbf{44, 44, 44,}\quad \overset{Q_2}{\mathbf{43,}}\quad \mathbf{42, 38, 38,}\quad \overset{Q_3}{\mathbf{31,}}\quad \mathbf{29, 27, 22}$$

The interquartile range is the spread between the first and third quartiles. That's the spread for the middle half of the data, since Q_1 is at the bottom of the second fourth of the data and Q_3 is at the top of the third fourth. For our family incomes, the interquartile range is $45-31$ or 14. Once again, the advantage of using the *IQR* is that it's not distorted by outliers. If Greer's income were 12,000 instead of 61, the *IQR* wouldn't change at all.

Standard Deviation

An even more accurate measure of the spread of a group of data is the standard deviation. This measure of spread is used extensively. To show you how it works, let's calculate the standard deviation of the simple group of data below.

2, 6, 2, 4, 3, 1

The first step is to calculate the mean of the data.

$$\bar{x} = \frac{2+6+2+4+3+1}{6} = 3$$

Next, we find the "deviation" of the data from the mean. That's just the difference between each of the numbers and the mean. The distance between the mean and the first number, 2, is 1. The difference between the second number, 6, and the mean is 3, and so on. Here are all the deviations.

1, 3, 1, 1, 0, 2

Notice we haven't bothered indicating whether the difference between each number and the mean is positive or negative. The reason is that in the next step, we're going to square each of these deviations. The square step eliminates any negatives anyway.

1, 9, 1, 1, 0, 4

Now we calculate the mean of the squared deviations.

$$\bar{x} = \frac{1+9+1+1+0+4}{6} \approx 2.67$$

The final step is to take the square root of 2.67.

$$\sigma = \sqrt{2.67} = 1.63$$

The number 1.63 is the standard deviation of our original data (2, 6, 2, 4, 3, 1). The symbol σ is the lower case sigma, which stands for standard deviation. This number is a measure of how far the numbers in the data are spread away from the mean. If the original data had been $0,\ 6,\ 2,\ 6,\ 3,\ 1$, the mean would have been the same (3). But since these numbers are more spread out from the mean, the standard deviation would have been 2.3, which is quite a bit higher. The only bad thing about the standard deviation is that it becomes distorted by outliers. But for data without outliers, it's more accurate than *IQR*. Here is the formula for calculating the standard deviation of a group of data.

Table 84.3

Formula for Standard Deviation	$\sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2}$

This looks complicated, but it's just a compact way of writing the steps that we just did. The term x_i represents each of the numbers in the data, and $\bar{x}$ is the mean. So $(x_i-\bar{x})^2$ is the square of the deviation between one number x_i

and the mean. And $\sum_{i=1}^{n}(x_i - \bar{x})^2$ is the square of all of the deviations. Multiplying this by $\frac{1}{n}$ is the same as dividing by n. That's how we calculate the mean of the square deviations. The last step is to take the square root of the whole thing. Here are the steps for calculating the standard deviation written individually.

Table 84.4

Individual steps for calculating standard deviation.	
1.	Calculate the mean of the data
2.	Find the deviation of each number from the mean
3.	Square each deviation
4.	Calculate the mean of the squared deviations
5.	Take the square root of this mean

Frequency Distributions

Even the standard deviation won't tell us everything about the spread of a group of data. To learn even more, we can look at a **frequency distribution** of the data. That's a list of how many times all the different numbers in the data appear. The two groups of data below have the same standard deviation. Yet the frequency distributions are quite different.

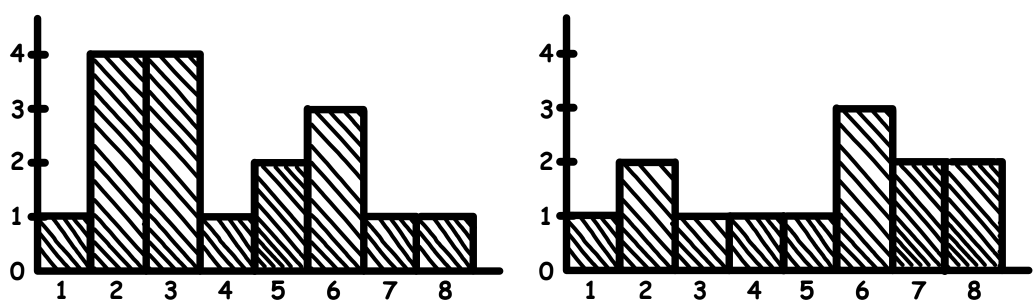
1,2,2,2,2,3,3,3,3,4,5,5,6,6,6,7,8

1,2,2,3,4,5,6,6,6,7,7,8,8

Notice that in the first group, the lower half has more numbers than the upper half. But in the second group of data, it's the other way around: The upper half has more numbers. The frequency distributions show this difference, but the standard deviations don't. (They're the same for both.)

A frequency distribution can also be put into a graph called a histogram. Each bar represents a range of the data. The horizontal axis measures the ranges and the vertical axis measures how many numbers in the data fall in each range. The taller the bar, the more numbers are in that particular range.

Figure 84.1

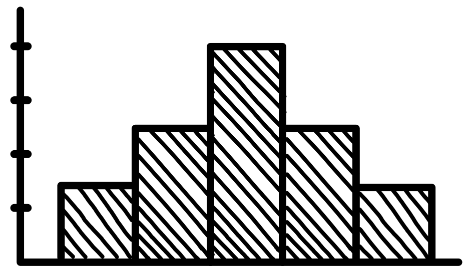


The nice thing about a histogram is that, with one glance, we can see the complete distribution. For instance, it's clear from the histograms above that both groups of data are "skewed." That just means the data has more numbers on one side than the other.

Normal Curve

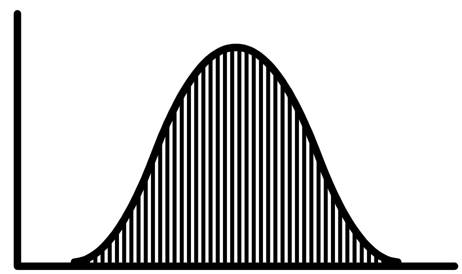
Not all data have skewed distributions. In fact, the most common frequency distribution is symmetrical. The most numbers are in the middle and then the bars get smaller as you go out. But everything is symmetrical around the center.

Figure 84.2



A symmetrical distribution of this type is called a **normal distribution**. If we narrow the ranges on the horizontal axis, the bars blend into a smooth curve.

Figure 84.3



This is called a normal curve or bell curve (because of its shape), and it's very famous in mathematics and science. The axis of symmetry of the normal curve is the mean of the data. The numbers to the right are above the mean, and those to the left are below the mean. Normal curves come up all the time in science. For instance, if a scientist conducts an experiment repeatedly, measuring his results each time, the distribution of those measurements is almost always normal. The differences in the measurements are due to error. Maybe the scientist's equipment can't measure the results that accurately. The scientist usually concludes that the true measurement in the experiment is the mean value, and all of the other measurements are off a bit. But what's interesting is that the errors fall symmetrically on either side of the mean. This happens over and over again when scientists conduct experiments.

In fact, normal curves appear in almost any physical situation. For instance, the distribution of the heights of all the men in a particular country is normal. There's a mean height, and then the men who are taller and shorter fall symmetrically around that mean. The distribution of the heights of women is also normal. So are the distributions of weights of men and of women. The distribution of IQ in a group of people turns out to be normal as well. There's a mean IQ and then, rising above and below the mean, the number of people gets smaller in a symmetrical pattern on either side.

Of course, not every one of the curves in each of these situations is exactly the same. The means and standard deviations are all different. The bigger the standard deviation, the fatter the curve will be. What's key is the overall shape of each curve (which is like a bell) and that the curve is symmetrical around the mean. Normal curves are actually a family of curves, just like parabolas are all part of a family. Even though specific parabolas are different, they all share the same basic characteristics and their equations all have the same general form. Normal curves are all exponential functions of the form $y = ae^{-\frac{1}{2}x^2}$. Of course, the general form can be transformed in various ways to fit specific distributions.

The 68%-95%-99.7% Rule

Probably the most important characteristic of the normal curve is its relationship to the standard deviation of the group of data the curve represents. No matter what the shape of a specific normal curve, 68% of the numbers in the data fall within 1 standard deviation of the mean, 95% fall within 2 standard deviations of the mean, and 99.7% fall within 3 standard deviations of the mean.

Table 84.5

For any normal curve
68% of data fall within 1 standard deviation of mean
95% fall within 2 standard deviations of mean
99.7% fall within 3 standard deviations of mean

For instance, if the mean of a group of data happens to be 22 and the standard deviation is 3.5, then 68% of all the data will fall within the range of 22 ± 3.5 or between 25.5 and 19.5. And 95% will fall within the range of 22 ± 7 or between 29 and 15. Finally, 99.7% will fall within the range of 22 ± 10.5 or between 32.5 and 11.5.

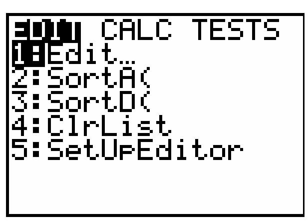
Table 84.6

Mean ($\bar{x}$)=22	Standard deviation (σ)=3.5
68%	22 ± 3.5 or between 25.5 and 18.5
95%	22 ± 7 or between 29 and 15
99.7%	22 ± 10.5 or between 32.5 and 11.5

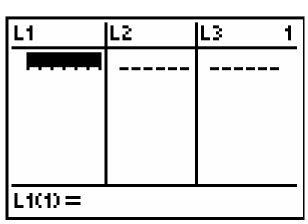
The bigger or smaller the standard deviation, the farther or closer the numbers fall from the mean. But these percentages will remain the same.

Standard Deviation on the Calculator

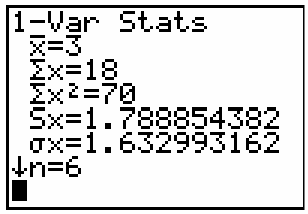
There's nothing complicated about figuring out the standard deviation of a group of data. But the process is kind of tedious. That's why most people calculate standard deviation using a calculator. The calculator can also be used to figure out the mean of the data. Let's go through the calculator steps to find the mean and standard deviation of the same data we worked with above: 2, 6, 2, 4, 3, 1. The first step is to press the STAT button, which stands for "statistics." That will give us the screen below.



Notice there are three modes to be in: EDIT, CALC, or TESTS. We're currently in EDIT mode, which is where we want to be. The next step is to select choice 1, which is Edit. That brings us to a series of columns for putting in data.



The cursor is currently in the first column, so we just start entering in our data. We press 2, then ENTER ; 6, then ENTER ; 2, then ENTER , and so on, until all of our data is in the first column. Now we press STAT again, to get back to the original menu. This time, we need to move from EDIT mode to CALC mode, by pressing the right arrow and then ENTER . This brings up "1-Var Stats," a function for calculating statistics. To make the function work, we need to press ENTER one more time. That brings up the screen below.



The first number is the mean, which is 3. We know that's right from when we calculated it by hand. The standard deviation is the number "σx ." It shows 1.63299…, which is also correct.

Populations vs. Samples

In addition to "σx ," the calculator screen also has a number "Sx," which, in our example, equals 1.78885…. This is another measure of standard deviation. To understand the difference between σx and Sx, we need to talk a little about populations and samples. A **population** is all of the numbers in a particular group of data. If we were doing statistical analysis on the state of Illinois, then the population would be all of the citizens of Illinois. It's often not practical to work with such large numbers, though. For instance, what if we were trying to predict the outcome of an election for governor of Idaho? It would probably be too expensive to ask every citizen of Idaho (the entire population) how they planned to vote. That would require hiring too many employees. They would have to drive all over the entire state, staying in hotels, etc.

A better way to predict the election would be to take a sample. A **sample** is just a smaller group taken from the entire population. So instead of asking everybody about their voting plans, we could ask a smaller number of Illinois citizens—maybe 1,000 people. If 700 people in our sample said that they planned to vote for the democrat candidate, then we could predict that the democrat would win the election. The only problem with a sample is the people selected might not represent the entire state. For instance, what if we had drawn our sample of people from a neighborhood with an unusually high number of democrats? That might be the reason 70% of the people supported the democrat. The actual election might turn out to be much closer, if the citizens of the entire state were more evenly balanced between democrats and republicans. To avoid these kinds of errors, a sample should be as similar to the entire population as possible. There are methods for ensuring this. In a "random" sample, the objects chosen should fairly represent the larger population.

Now let's go back to the difference between σx and Sx. The σx standard deviation, which is called the "sigma formula," assumes that our data includes the entire population. The Sx standard deviation, called the "*s* formula," assumes that we're calculating the standard deviation of a sample. The numbers are different, because the calculator adjusts for the fact that the sample might not perfectly represent the entire population. So in doing your own calculations, use the sigma formula when the data is the complete population, and the *s* formula when the data is just a sample.

Practice 84

a. If a die is rolled 2 times, what is the probability of not rolling a 3 either time? Write your answer as a fraction.

b. There are two bags. The first bag has 7 green and 5 yellow marbles. The second bag has 3 green and 4 yellow marbles. What is the probability of choosing a yellow marble without looking? Write your answer as a fraction.

c. The salaries of 15 employees at the Gamma Corporation were recorded as follows: \$45,000, \$62,500, \$53,000, \$74,500, \$50,700, \$68,400, \$70,800, \$53,000, \$48,300, \$52,100, \$47,000, \$57,600, \$56,300, \$65,700, \$53,000.

Find the standard deviation of the salaries. Estimate your answer to 2 decimal places.

d. A set of data with 400 numbers is normally distributed with a mean of 36 and a standard deviation of 5. Within what range will 99.7% of the data fall?

A. from 0 to 36　　B. from 0 to 72　　C. from 31 to 41
D. from 26 to 46　　E. from 21 to 51

e. The mean score on a chemistry test is 70. If the teacher decides to scale the grades by increasing each score by 6 points, what is the new mean of the data?

f. Solve the word problem below.

A bell curve fits data so that only 4% of the overall sample falls outside of 2 standard deviations (in either direction) from the mean. During Caleb's last history test, the standard deviation of the scores was 11.1 points and the mean was 67 points. Given this information, what was the range of scores for 96% of the class?

Problem Set 84

Tell whether each sentence below is True or False.

1. Standard deviation is a popular (and accurate) measure of the spread or distribution of a group of data.

2. For data that is normally distributed, 68% of the numbers in the data will fall within 1 standard deviation of the mean, 95% will fall within 2 standard deviations of the mean, and 99.7% will fall within 3 standard deviations of the mean.

Answer each question below.

3. Find the 31st term in the arithmetic sequence with $a_1 = 3$ and a common difference (d) of 7.

4. Find the common ratio (r) in the following geometric sequence: $9,\ 6,\ 4,\ \frac{8}{3},\ \frac{16}{8}\ ...$

5. If the common ratio (r) of a geometric sequence is 4 and $a_4 = 32$, find the first term of the sequence.

Calculate the number of permutations in each problem below.

6. How many possible ways can 8 people be arranged to sit at a conference table with 8 chairs?

7. How many different ways can the letters of the word BOOTH be arranged?

Answer each question below.

8. Four machines are used in the assembly line of a factory. There is a 0.15 probability that each machine will fail due to thermal problems. What is the probability that all 4 machines will fail due to thermal problems? Estimate your answer to 4 decimal places.

9. A 5 member board of directors is being selected from a list of 15 candidates. How many different board member combinations are possible?

10. The probability that Aaron will win the prize is $\frac{3}{5}$, and, independently, the probability that Carrie will win the prize is $\frac{7}{12}$. What is the probability that Carrie will win the prize and that Aaron will not? Write your answer as a fraction.

(a) 11. If a die is rolled 2 times, what is the probability of not rolling a 4 either time? Write your answer as a fraction.

12. There are 8 strawberry smoothies, 4 banana smoothies, and 6 pineapple smoothies in a refrigerator. What is the probability that James (without looking) will pull out 2 strawberry smoothies? Write your answer as a fraction.

(b) 13. There are two bags. The first bag has 4 blue and 6 red pencils. The second bag has 5 blue and 2 red pencils. What is the probability of choosing a red pencil without looking? Write your answer as a fraction.

Answer each question below.

The freshman enrollment for the year 2007 in a population of 15 colleges was recorded as follows: 6,263, 4,078, 9,351, 5,446, 5,004, 4,652, 4,972, 4,483, 6,866, 6,260, 7,962, 7,278, 8,760, 4,843, 5,035.

14. Find the range and interquartile range (*IQR*) of the college freshman enrollments. Range=5,273,

(c) 15. Find the standard deviation of the college freshman enrollments. Estimate your answer to 2 decimal places.

Select the answer of each question below.

16. A set of data with 100 numbers is normally distributed with a mean of 28 and a standard deviation of 5. What percent of the data will fall within the range from 18 to 38?

A. 5% B. 68% C. 50%
D. 95% E. 99.7%

(d) 17. A set of data with 500 numbers is normally distributed with a mean of 45 and a standard deviation of 12. Within what range will 68% of the data fall?

A. from 0 to 45 B. from 0 to 90 C. from 33 to 57
D. from 21 to 69 E. from 9 to 81

Select the answer for the following question.

18. What does the angle $420°$ equal in radians?

A. $\frac{2\pi}{3}$ B. $\frac{4\pi}{3}$ C. $\frac{7\pi}{3}$

D. $\frac{16\pi}{3}$ E. $\frac{7\pi}{6}$

Answer each question below.

19. A pair of coins was tossed 14 times and the number of tails per toss was as follows: {2, 1, 0, 1, 0, 2, 0, 0, 1, 0, 1, 0, 1, 2}. What is the mean of the data?

20. Matt's average score on the first four tests of his College Algebra course was 83. If he makes a 90 on the fifth test, what will his new test average be?

(e) 21. The mean score on a math test is 75. If the teacher decides to scale the grades by increasing each score by 5 points, what is the new mean of the data?

Solve the word problem below.

(f) 22. A bell curve fits data so that only 4% of the overall sample falls outside of 2 standard deviations (in either direction) from the mean. In a certain experiment on life expectancy, the standard deviation of the data was 5.7 years and the mean was 72 years. Given this information, what is the range of life expectancy for 96% of the population?