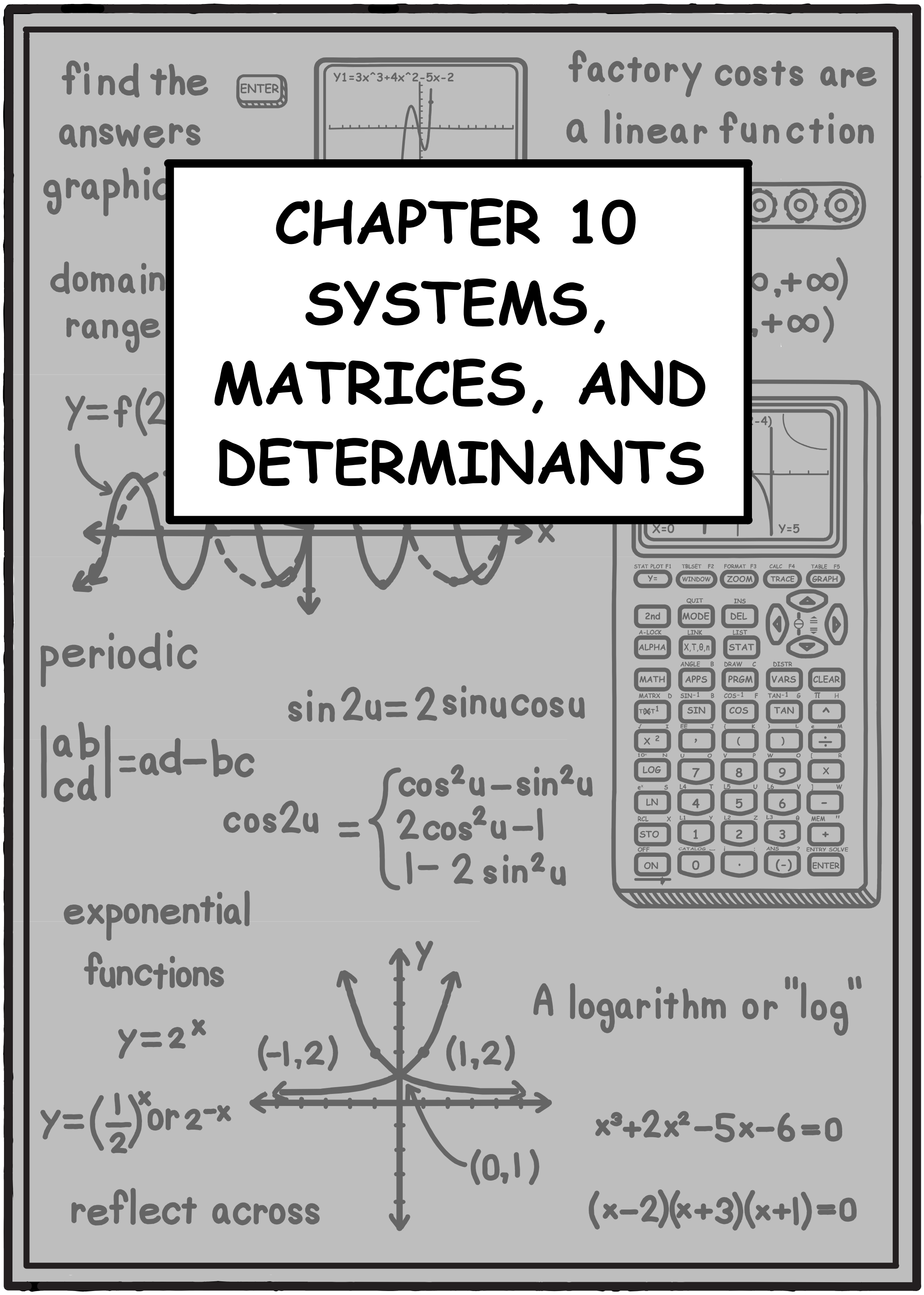
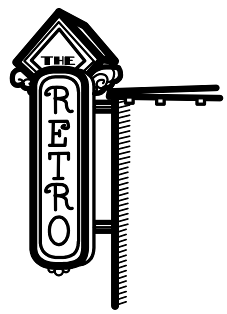
CHAPTER 10 SYSTEMS, MATRICES, AND DETERMINANTS



Lesson 64—Solving Systems

In this chapter, we're going to focus on systems of equations. As you may remember from algebra, systems are used to solve problems that have more than one unknown. Let's go through a fairly simple example.



The Retro Theater charges \$8 for an adult ticket and \$5 for a child's ticket. Last Saturday, Retro sold 405 tickets for a total of \$2,658. How many adults and how many children bought tickets last Saturday?

Notice there are two unknowns that we're trying to find: The number of adults and the number of tickets. This is the kind of problem where a system is helpful. The first step is to let A be the number of adults who bought a ticket and C the number of children. That gives us two unknowns (or variables). The next step is to write not one but two equations, each with A and C. Since a total of 405 tickets were sold on Saturday, we know that the number of adults plus the number of children must equal 405. That translates into the equation $A+C=405$. To write a second equation, we can use the fact that the total amount of money collected was \$2,658. Since each adult ticket sold for \$8, the amount of money from adults must have been $8A$. And since each child's ticket sold for \$5, the amount of money from children must have been $5C$. That gives us the equation $8A+5C=2,658$. These two equations together make up the system below.

$$\begin{cases} A+C=405 & \textbf{Equation 1} \\ 8A+5C=2,658 & \textbf{Equation 2} \end{cases}$$

There are lots of pairs of values for A and C that will solve each equation individually. For instance, $A=1$, $B=404$ will solve Equation 1. But so will $A=202$, $B=203$ and $A=157$, $B=248$. And there are many more. The same is true for Equation 2. There are many pairs that will work. You can find some yourself. But the important point is that only one pair will solve *both* equations. And that pair is the solution to the system.

So how do we find the right pair? There are different methods for solving a system. You may remember them from algebra. To solve with the substitution method, we need to first solve Equation 1 for one of the variables. Let's solve for A.

$$\begin{cases} A=405-C & \textbf{Equation 1} \\ 8A+5C=2,658 & \textbf{Equation 2} \end{cases}$$

Equation 1 says that A is the same as $405-C$, which means that we can substitute that expression for A in Equation 2.

$$8(405-C)+5C=2,658$$

The substitution step creates an equation with just one unknown, *C*. Now we can solve for *C*.

$$3,240 - 8C + 5C = 2,658$$

$$-3C = -582$$

$$C = 194$$

So *C* must equal 194. With that information, it's easy to solve for *A*. All we have to do is go back to either one of the original equations and put 194 in for *C*. We'll use Equation 1 since it's easier.

$$A + C = 405$$

$$A + 194 = 405 \qquad \textbf{substituting}$$

$$A = 211$$

We end up with $A = 211$, $C = 194$, which means that Retro must have sold tickets to 211 adults and 194 children. You can check these answers yourself by substituting them back into the two original equations. You'll find that $A = 211$, $C = 194$ solves both.

Elimination Method

Instead of substituting, we could also have solved our system by either addition or subtraction. To show you how it works, we'll solve the system again, this time by using addition. The first step is to multiply both sides of Equation 1 by -8.

$$\begin{cases} -8(A + C) = (-8)(405) & \textbf{Equation 1} \\ 8A + 5C = 2,658 & \textbf{Equation 2} \end{cases}$$

Now we simplify.

$$\begin{cases} -8A + (-8C) = -3,240 & \textbf{Equation 1} \\ \ \ 8A \ + \ 5C \ = \ 2,658 & \textbf{Equation 2} \end{cases}$$

Notice that the *A* terms are opposites. That means if we add the left and right sides of Equation 1 to the same sides of Equation 2, the *A*'s will cancel out. It's perfectly legal to do this addition step, by the way. Since the expressions in Equation 1 are equal, we're just adding the same quantity to both sides of Equation 2.

$$-8A + (-8C) + 8A + 5C = 2,658 + (-3,240)$$

The next step is to simplify by combining like terms and adding numbers.

$$-8C + 5C = -582$$

Notice that since the *A*'s were opposites, they canceled each other out. We're left with an equation that has just one unknown (*C*). Finishing the steps, we get the same answer for C as before.

$$-3C = -582$$

$$C = 194$$

Now all we would have to do is substitute 194 for C in either one of the original equations and solve for A. The steps are exactly the same as before, so we won't bother showing them again. The main point is that instead of substituting to get one of the equations down to a single variable, we can also add the expressions of one equation to both sides of the other equation. But to use the addition method, one pair of terms must be opposites. That's the only way they'll cancel each other out, leaving just the other variable.

There's also such a thing as a subtraction method. If we had multiplied both sides of Equation 1 by 8 instead of -8, the A terms would have been exactly the same.

$$\begin{cases} 8(A+C)=(8)(405) & \textbf{Equation 1} \\ 8A+5C=2,658 & \textbf{Equation 2} \end{cases}$$

$$\begin{cases} 8A+8C=3,240 & \textbf{Equation 1} \\ 8A+5C=2,658 & \textbf{Equation 2} \end{cases}$$

We can still get the A terms to cancel out. But we have to do it by *subtracting* both sides of Equation 1 from both sides of Equation 2.

$$8A+5C-(8A+8C)=2,658-3,240$$

$$8A+5C-8A-8C=2,658-3,240$$

$$-3C=-582$$

$$C=194$$

See, we end up with the same answer for C as before. The main point is that another way to eliminate a variable in a system is to add or subtract both sides of one equation to (from) the other. These two methods together are sometimes called the elimination method.

Multiplying Both Equations

In some systems, both equations need to be multiplied by something in order to eliminate a variable. Here's an example like that.

$$\begin{cases} 2x+3y=-4 & \textbf{Equation 1} \\ 3x-5y=-25 & \textbf{Equation 2} \end{cases}$$

There's no way we can make either the x's or the y's the same or opposites by multiplying just one of the equations. What we can do, though, is multiply both sides of Equation 1 by -3 and both sides of Equation 2 by 2. This will make the x terms opposites.

$$\begin{cases} -3(2x+3y)=-3(-4) & \textbf{Equation 1} \\ 2(3x-5y)=2(-25) & \textbf{Equation 2} \end{cases}$$

Simplifying gives us this.

$$\begin{cases} -6x-9y=12 & \textbf{Equation 1} \\ 6x-10y=-50 & \textbf{Equation 2} \end{cases}$$

See, the x's are opposites, which means we can eliminate those terms by adding both sides of Equation 1 to both sides of Equation 2.[1]

$$-6x-9y+6x-10y=-50+12$$

$$-9y-10y=-38$$

Now we have just one variable in the equation, so we can solve for y.

$$-19y=-38$$

$$y=2$$

To find x, we just need to substitute 2 for y in either one of the original equations. Let's use Equation 1.

$$-6x-9y=12$$

$$-6x-9(2)=12$$

$$-6x-18=12$$

$$-6x=30$$

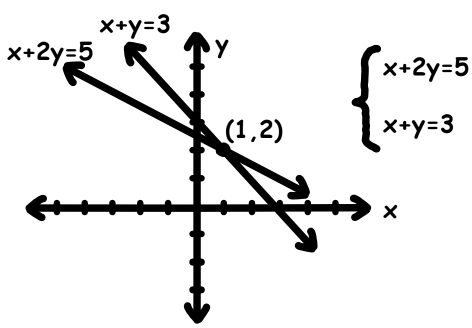
$$x=-5$$

The solutions to the system are $x=-5$, $y=2$. And you can check that for yourself by substituting this pair in for both or the original equations.

Solving Systems by Graphing

Another way to solve a system is by graphing both equations on the same coordinate plane. When both of the equations are linear, as with all of our examples, the graph will be two straight lines. Here's the graph of a linear system.

Figure 64.1



Notice that the two lines intersect at just one point: $x=1$, $y=2$. That's the solution pair that will solve both equations. Remember, all the points on each line will solve the equation for that line, but only the point that's on both lines will solve both equations. To solve a system by graphing, then, you have to graph each equation on the same coordinate plane and look for the point(s) where the graphs intersect.

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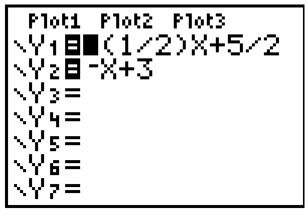
[1] Adding both sides of Equation 2 to Equation 1 would have worked just as well.

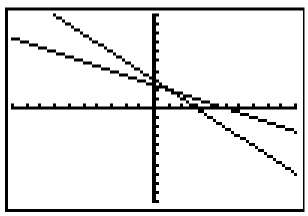
If the coordinates of the intersection point are fractions or decimals, then it can be hard to see the exact answers to the system. One way to get around this problem is to graph the equations on your calculator. To graph the system $x+2y=5$ and $x+y=3$ this way, the first step is to solve both equations for *y*. That gives us this.

$$\begin{cases} y=-\dfrac{1}{2}x+\dfrac{5}{2} \\ y=-x+3 \end{cases}$$

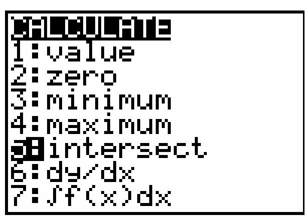
Next, we enter both of these into the calculator in the usual way.



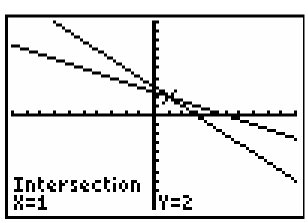
The standard window ($\text{Xmin}=-10$, $\text{Xmax}=10$, $\text{Ymin}=-10$, $\text{Ymax}=10$) will work fine for this system. So to see the graph of both functions on the same plane, we can just press GRAPH .



Now to find the coordinates of the intersection point, we use the "calculate" function. We just press 2nd , then TRACE to get to that menu. We need "intersect," which is choice 5.



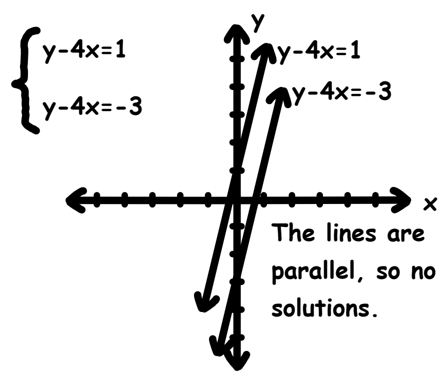
Now we select the equation $y=-\dfrac{1}{2}x+\dfrac{5}{2}$ as the "first curve," and $y=-x+3$ as the "second curve." The last step is to "guess," by moving the cursor onto the intersection point. The answer comes out to $x=1$, $y=2$, which we know is right. The main point, though, is that graphing a system on a calculator can be helpful when the coordinates are fractions or decimals.



Some Special Cases

Some systems are a special case because they don't have any solutions. The easiest way to see why is to graph the two equations. Look at the graph of this system.

Figure 64.2



Since the lines are parallel, they never intersect. That means the system has no solutions. If we were to try to solve the system using algebra, with the substitution or elimination method, we would end up with a false equation. Watch this.

$$\begin{cases} y-4x=1 & \textbf{Equation 1} \\ y-4x=-3 & \textbf{Equation 2} \end{cases}$$

Subtracting Equation 2 from Equation 1 gives us an obviously false equation.

$$y-4x-(y-4x)=1-(-3)$$

$$0=4$$

That's always what happens when a linear system has no solutions. Both variables will cancel out, and you'll be left with a false equation. Systems with no solutions are said to be **inconsistent**.

There's one other special case we should review. That's when the two equations are actually the same. Here's a case like that.

$$\begin{cases} y-3x=5 & \textbf{Equation 1} \\ 2y=6x+10 & \textbf{Equation 2} \end{cases}$$

It's kind of hard to tell that these equations are the same, because they're in a different form. But watch what happens when we divide both sides of Equation 2 by 2 and move the x term to the left side.

$$\frac{2y}{2}=\frac{6x+10}{2}$$

$$y=\frac{2(3x+5)}{2}$$

$$y=3x+5$$

$$y-3x=5$$

Equation 2 is exactly the same as Equation 1. A system like this actually has an infinite number of solutions, because any linear equation has an infinite number of solution pairs. Since both equations are the same, all those pairs will solve the system. If we had tried to solve this algebraically, both the *x*- and *y*-terms would have canceled out and we would have been left with a true equation like $0=0$. That always happens when both equations of a system are actually the same. Systems of this kind are called **dependent** systems.

Practice 64

a. Select the choice that is identical to each trig expression: $\dfrac{1+\csc x}{\cot x+\cos x}$

A. $\csc x$ B. $\cos x$ C. $\sec x$
D. 1 E. $\sin x$

b. Use your knowledge of trig identities (not a calculator) to find the trig value below.

If $\cos 2\theta=\dfrac{3}{8}$, find $\dfrac{1}{\cos^2\theta-\sin^2\theta}$

c. Solve each system of equations: $\begin{cases}-5x+7y=31\\3x+4y=6\end{cases}$

d. Solve the system of equation $\begin{cases}2x+3y=10\\3x-y=-7\end{cases}$ using a graphing calculator.

e. The graph of $y=-4x+7$ can be expressed as a set of parametric equations. If $x=1-2t$, and $y=f(t)$, then what does $f(t)$ equal?

A. $\dfrac{1}{2}t-\dfrac{5}{2}$ B. $2t+3$ C. $8t-13$
D. $-\dfrac{1}{4}t-\dfrac{7}{4}$ E. $8t+3$

f. Solve the word problem below.

A local convenience store recently ran a sale on their fountain drinks. During the sale, a large fountain drink cost \$1.50 and a small cost \$1.00. If 501 drinks were sold altogether and the store made \$689 during the sale, how many of each size were sold?

Problem Set 64

Tell whether each sentence below is True or False.

1. Systems are used to solve problems that have more than one unknown.

2. Systems can be solved by the substitution, elimination, or graphing methods.

For each pair of polar coordinates below, select the correct conversion into rectangular coordinates. Don't use the polar conversion function on your calculator.

3. $P(170, 135°)$

A. $(85\sqrt{2}, 85\sqrt{2})$ B. $(-85\sqrt{2}, 85\sqrt{2})$ C. $(-85\sqrt{3}, -85)$
D. $(85, -85\sqrt{3})$ E. $(85\sqrt{2}, -85\sqrt{2})$

4. $Q(50, -60°)$

A. $(-25\sqrt{3}, 25)$ B. $(-25, -25\sqrt{3})$ C. $(-25, 25\sqrt{3})$
D. $(25, -25\sqrt{3})$ E. $(25\sqrt{2}, -25\sqrt{2})$

For each pair of rectangular coordinates below, select the correct conversion into polar coordinates. Don't use the polar conversion function on your calculator.

5. $M(8, 6)$

A. $(-10, 36.9°)$ B. $(13, -36.9°)$ C. $(10, -36.9°)$
D. $(-10, -36.9°)$ E. $(10, 36.9°)$

6. $N(-24, 9)$

A. $(-25.6, -159.4°)$ B. $(-15, -20.6°)$ C. $(25.6, 159.4°)$
D. $(-25.6, 159.4°)$ E. $(25.6, -159.4°)$

Select the choice that is identical to each trig expression given.

7. $\dfrac{\cos^2\theta}{1-\sin\theta} - 1$

A. $\sin\theta$ B. $\cos\theta$ C. $\tan\theta$
D. $\sec\theta$ E. $\csc\theta$

(a) 8. $\dfrac{1+\sec x}{\tan x + \sin x}$

A. 1 B. $\sec x$ C. $\sin x$
D. $\csc x$ E. $\cos x$

Use your knowledge of trig identities (not a calculator) to find each trig value below.

9. If $\tan(-\theta) = -4$, find $\cot\theta$

(b) 10. If $\cos 2\alpha = \dfrac{6}{7}$, find $\dfrac{1}{\cos^2\alpha - \sin^2\alpha}$

Solve each system of equations below.

11. $\begin{cases} x+3y=18 \\ -x+2y=7 \end{cases}$

12. $\begin{cases} x+y=-2 \\ 3x-y=10 \end{cases}$

(c) 13. $\begin{cases} -3x+10y=5 \\ 2x+7y=24 \end{cases}$

Solve each system of equations below using a graphing calculator.

14. $\begin{cases} 5x-y=16 \\ 2x+3y=3 \end{cases}$

(d) 15. $\begin{cases} x-2y=-8 \\ 2x-y=-7 \end{cases}$

Select the answer for each question below.

16. The graph of $y=-x^4+9x-12$. (Hint: Use your graphing calculator.)

A. intersects the x-axis at exactly one point.
B. intersects the x-axis at exactly two points.
C. intersects the x-axis at exactly three points.
D. intersects the x-axis at exactly four points.
E. does not intersect the x-axis.

17. Which of the following is equivalent to the equation $x^2+7x+y^2-2=0$ in polar form?

A. $r^2=7r\cos\theta$ B. $r=7r\cos\theta-2$ C. $r^2=-7r\sin\theta$
D. $r^2+7r\sin\theta-2=0$ E. $r^2+7r\cos\theta-2=0$

(e) 18. The graph of $y=-3x+4$ can be expressed as a set of parametric equations. If $x=1-t$, and $y=f(t)$, then what does $f(t)$ equal?

A. $4t-5$ B. $3t+1$ C. $\frac{1}{3}t+1$
D. $-\frac{1}{3}t-4$ E. $-3t+1$

Answer each question below. Estimate any irrational answers to two decimal places.

19. If $x=4\cos\theta$ and $y=4\sin\theta$, then $\sqrt{x^2+y^2}=?$

20. What is the magnitude of vector $\mathbf{v}$ with initial point $(2,-7)$ and terminal point $(-4,5)$?

21. If $12\sin^2\alpha-4\sin\alpha-1=0$ over the interval $0°\le\alpha\le180°$, then $\alpha=?$ (All angles are in degrees.)

Solve the word problem below.

(f) 22. City of Fun, an amusement park, charges \$25 per child and \$45 per adult. On an average summer day, they sell a total of 2,210 tickets and make \$75,450. Based on this data, how many of each ticket type do they sell on an average summer day?

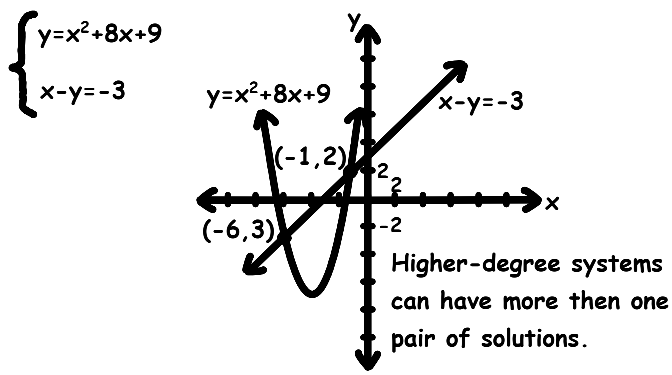
Lesson 65—More Complicated Systems

We started learning about systems in the last lesson. All of our examples involved two linear equations. But, as you probably remember from algebra, some systems have higher-degree equations. Here's an example.

$$\begin{cases} y = x^2 + 8x + 9 & \textbf{Equation 1} \\ x - y = -3 & \textbf{Equation 2} \end{cases}$$

Equation 1 is a parabola, and Equation 2 is a line. A graph of this system shows that the graphs intersect at not one but two points.

Figure 65.1



This is one of the big differences between linear systems and higher-degree or "non-linear" systems. When non-linear equations are involved, it's possible for the graphs to intersect more than once. That makes the system have more than one pair of solutions.

We can find both solution pairs without ever looking at the graph. We can solve the system algebraically by the same basic method we use for linear systems. Since Equation 1 is already solved for *y*, the substitution method is probably the easiest. So we substitute $x^2 + 8x + 9$ for *y* in Equation 2 and then simplify.

$$x - (x^2 + 8x + 9) = -3$$

$$x - x^2 - 8x - 9 = -3$$

$$x = -3 + x^2 + 8x + 9$$

$$0 = x^2 + 7x + 6$$

We end up with a quadratic equation with just one variable *x*. The equation can be solved by factoring.

$$0 = (x + 6)(x + 1)$$

$$x = -6\,,\ -1$$

As with most quadratic equations, we end up with two solutions. We're not finished yet, though, because we still have to find matching values for *y*. What we do is substitute each of our answers for *x* into either of the original equations. Equation 2 is a lot simpler, so we'll use it.

for x=-6	**for x=-1**
$-6-y=-3$	$-1-y=-3$
$y=-3$	$y=2$

We end up with the solution pairs $(-6,-3)$ and $(-1,2)$. And those were the intersection points on the system's graph. That's how a non-linear system can be solved algebraically.

Here's another example of a non-linear system. This one has a third-degree (cubic) function.

$$\begin{cases} y = x^3 - x^2 & \textbf{Equation 1} \\ y = 2x^2 & \textbf{Equation 2} \end{cases}$$

See, Equation 1 is third degree and Equation 2 is second degree. To solve the system algebraically, we should use the substitution method, since both equations are solved for *y*. Let's substitute $2x^2$ for *y* in Equation 1. That will eliminate *y*.

$$2x^2 = x^3 - x^2$$

Now we move everything to the left and simplify.

$$3x^2 - x^3 = 0$$

This equation can be solved by factoring.

$$x^2(3-x) = 0$$

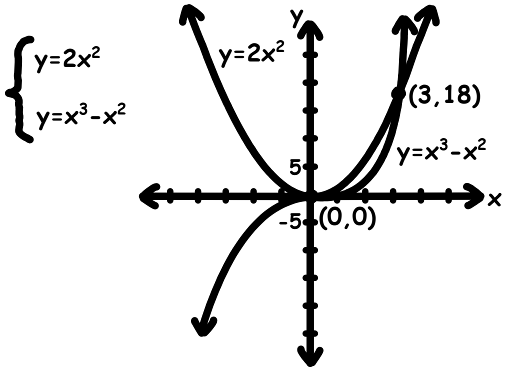
$x^2 = 0$	$3-x=0$
$x=0$	$x=3$

We end up with two answers for *x*. To find the matching values for *y*, we substitute these into either one of the original equations. Equation 2 is simpler.

for x=0	**for x=3**
$y=2(0)^2$	$y=2(3)^2$
$y=0$	$y=18$

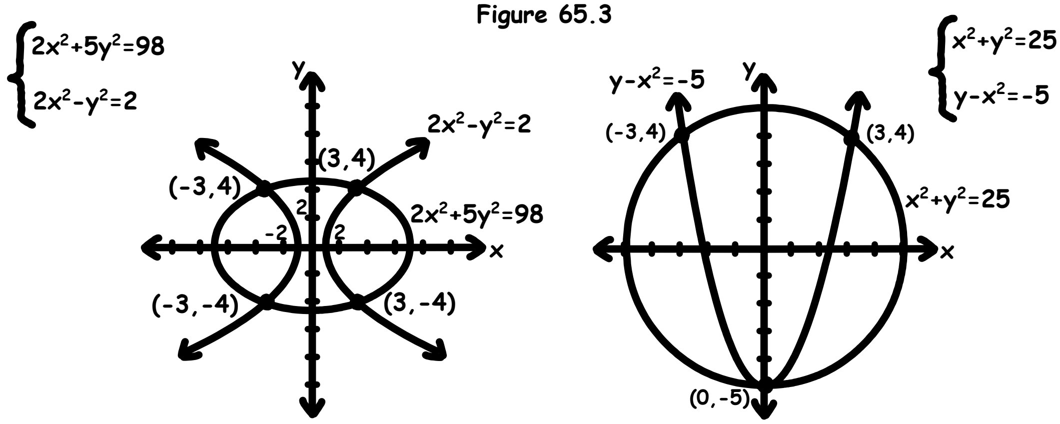
So this system also has two solution pairs: $(0,0)$ and $(3,18)$. The graph of the system confirms these answers.

Figure 65.2



Obviously, there are lots of other kinds of non-linear systems besides our two examples. Some have even more than two pairs of solutions. Here are a couple of other examples.

Figure 65.3



Some non-linear systems are so hard that they can't be solved with algebra. That's when the graphing calculator becomes essential. To find the solutions, you can use the "intersect" function, as we discussed in the last lesson.

Three Variables, Three Equations

Linear systems can also get more complicated. Instead of having just two equations with two variables, it's possible for a system to have three equations with three variables. As you may remember, this kind of system would be useful in solving a problem with three unknowns. Here's an example of a three-variable linear system.

$$\begin{cases} 5x-4y+3z=15 & \textbf{Equation 1} \\ 6x+2y+9z=13 & \textbf{Equation 2} \\ 7x+6y-6z=6 & \textbf{Equation 3} \end{cases}$$

We solve three-variable, three-equation systems using basically the same method as we do systems having only two variables. The first step is to eliminate a variable. We can make the y-terms in Equations 1 and 2 opposites by multiplying both sides of Equation 2 by 2.

$$2(6x+2y+9z)=2(13) \qquad \textbf{multiplying by 2}$$

That gives us this.

$$\begin{cases} 5x-4y+3z=15 & \qquad \textbf{Equation 1} \\ 12x+4y+18z=26 & \qquad \textbf{Equation 2} \\ 7x+6y-6z=6 & \qquad \textbf{Equation 3} \end{cases}$$

Now we can eliminate the y's in those two equations by adding them. We'll add the sides of Equation 1 to the sides of Equation 2.

$$12x+4y+18z+\overbrace{(5x-4y+3z)}^{\textbf{Added from Equation 1.}}=26+15$$

$$17x+21z=41$$

Instead of getting an answer for x or z, we end up with a two-variable equation. There's no way to pin down specific answers for either of those variables right now. So the next step is to go back to the original system and try to eliminate the y-terms in two other equations. We used Equations 1 and 2 before, so this time let's eliminate the y's in Equations 2 and 3. To make those terms opposites, we can multiply both sides of Equation 2 by -3.

$$-3(6x+2y+9z)=-3(13) \qquad \textbf{multiplying by 2}$$

That gives us this.

$$\begin{cases} 5x-4y+3z=15 & \qquad \textbf{Equation 1} \\ -18x-6y-27z=-39 & \qquad \textbf{Equation 2} \\ 7x+6y-6z=6 & \qquad \textbf{Equation 3} \end{cases}$$

Now we can add Equation 3 to Equation 2 and simplify.

$$-18x-6y-27z+\overbrace{(7x+6y-6z)}^{\textbf{Added from Equation 3.}}=-39+6$$

$$-11x-33z=-33$$

We end up with another two-variable equation. We can't get a specific value for x or y with this equation either. But what we can do is put together the two-variable equations that we've created in order to create another system. We'll call the first two-variable equation, $17x+21z=41$, Equation 4, and the second two-variable equation, $-11x-33z=-33$, Equation 5.

$$\begin{cases} 17x+21z=41 & \qquad \textbf{Equation 4} \\ -11x-33z=-33 & \qquad \textbf{Equation 5} \end{cases}$$

We already know how to solve two-variable, two-equation systems. So now we can find solutions for x and y. We can make the x's opposites by multiplying both sides of Equation 4 by 11 and both sides of Equation 5 by 17.

$$\begin{cases} 11(17x+21z)=11(41) & \textbf{Equation 4} \\ 17(-11x-33z)=17(-33) & \textbf{Equation 5} \end{cases}$$

$$\begin{cases} 187x+231z=451 & \textbf{Equation 4} \\ -187x-561z=-561 & \textbf{Equation 5} \end{cases}$$

Now we can eliminate the x's by adding and simplifying.

Added from Equation 4.

$$-187x - 561z + \overbrace{(187x + 231z)} = -561 + 451$$

$$-330z = -110$$

$$z = \frac{1}{3}$$

We have a value for z. To find a matching value for x, we just substitute $z = \frac{1}{3}$ into either one of the two-variable equations. Using Equation 4, we get this.

$$17x + 21\left(\frac{1}{3}\right) = 41$$

Now we solve for x.

$$17x + 7 = 41$$

$$17x = 34$$

$$x = 2$$

With values for both z and x, we can find the matching value for y by substituting into any of the original three-variable equations. Using Equation 2, we get this.

$$6(2) + 2y + 9\left(\frac{1}{3}\right) = 13$$

$$12 + 2y + 3 = 13$$

$$2y = -2$$

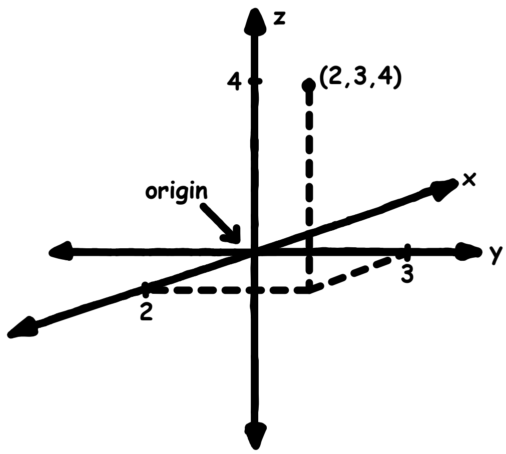
$$y = -1$$

The solutions to the system are $x = 2$, $y = -1$, and $z = \frac{1}{3}$.

Graphing in 3-D

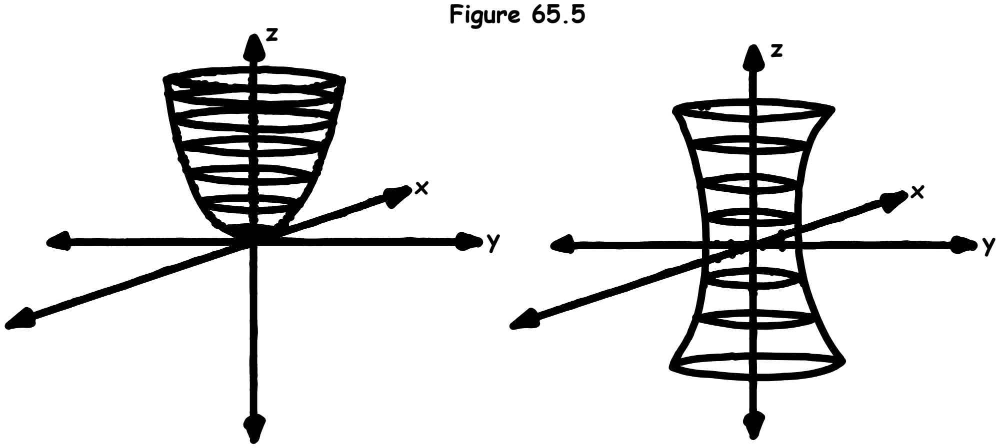
What does the graph of a three-variable, three-equation system look like? Well, you may remember from algebra that graphs of three-variable equations don't fit on a flat coordinate plane. They're 3-dimensional. The coordinate system has an *x*-axis, a *y*-axis and a *z*-axis, with the *z*-axis extending above and below the other two axes. Here's what the point $x = 2$, $y = 3$, $z = 4$ looks like in a 3-D coordinate system.

Figure 65.4



In a 3-D graph, every point in space is represented by not two but three numbers: (x, y, z). Graphs of three-variable equations are also 3-dimensional. They have all sorts of interesting shapes. Here are a few examples.

Figure 65.5



A linear three-variable equation has the simplest of all 3-D graphs. It's just a plane (like a flat piece of paper) floating in space. The graph of our system, then, is three planes in space. Our three answers ($x = 2$, $y = -1$, $z = \frac{1}{3}$) represent the point (in space) where all three of those intersect.

Practice 65

a. For the pair of rectangular coordinates $K(-35,-18)$, select the correct conversion into polar coordinates. Don't use the polar conversion function on your calculator.

A. $(39.4,-207.2°)$ B. $(-39.4,-207.2°)$ C. $(39.4,207.2°)$
D. $(-39.4,207.2°)$ E. $(-53,27.2°)$

Solve each system of equations below.

b. $\begin{cases} y = x^2 - 4x + 1 \\ x - y = 3 \end{cases}$

c. $\begin{cases} 3x - 2y + 7z = 30 \\ -x + 2y - 3z = -12 \\ x - y - z = 7 \end{cases}$

d. Solve the system of equations $\begin{cases} x^2 + y^2 = 16 \\ y = x^4 - 2x^2 + 3x + 2 \end{cases}$ using a graphing calculator. Estimate any irrational answers to two decimal places.

e. $f(x) = \sin(\arctan x)$ and $g(x) = \tan(\arcsin x)$. If $0 \le x \le \dfrac{\pi}{2}$, then $g\left(f\left(\dfrac{\pi}{12}\right)\right) = ?$

f. Solve the word problem below.

Big Time Cinema offers three different ticket prices. Children tickets cost $4.50, adult tickets cost $9.00, and senior tickets cost $5.50. On a certain Friday night, the theater sold twice as many adult tickets as it did senior and child tickets combined. If the theater sold a total of 2,121 tickets and made $16,158.50, how many of each ticket type did the theater sell?

Problem Set 65

Tell whether each sentence below is True or False.

1. Higher-degree systems can have more than one pair of solutions.

2. Graphs of three-variable, three-equation systems can be graphed on a flat coordinate plane.

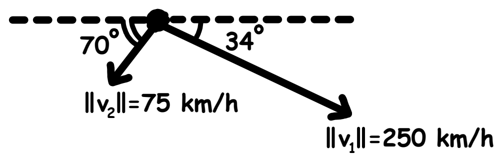
Answer each question below. Estimate any irrational numbers to one decimal place.

3. Find the components of a vector **h** with magnitude 15 and direction angle $\theta = -20°$.

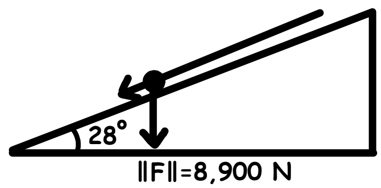
4. Find the magnitude and direction angle of $\mathbf{k} = -34\mathbf{i} - 7\mathbf{j}$.

Answer each question below. Estimate any irrational answers to one decimal place.

5. An airplane is flying 250 kilometers per hour and heading $34°$ south of east. If the wind is blowing at a rate of 75 kilometers per hour $70°$ south of west, what is the groundspeed and direction of the airplane?



6. Cars that slide down icy hills often have to be pulled up by tow trucks. If a small car weighing 8,900 Newtons is held still by a cable from a tow truck on an ice slide inclined at an angle of $28°$, what is the pull on the cable?



For each pair of parametric equations below, eliminate the parameter and select the direct relationship between x and y.

7. $x = -4 - \frac{1}{3}t$, $y = 2t$

A. $y = -8 - \frac{2}{3}x$ B. $y = -4 - \frac{2}{3}x$ C. $y = -4 - \frac{1}{6}x$

D. $y = -6x - 24$ E. $y = -8x - \frac{2}{3}x^2$

8. $x = 3t - 2$, $y = 18t^2 + 6t$

A. $y = 2x^2 + 10x + 12$ B. $y = 54x^2 + 18x - 2$ C. $y = \frac{2}{3} + \frac{1}{3}x$

D. $y = 162x^2 - 198x + 60$ E. $y = 54x^3 - 18x^2 - 12x$

For each pair of rectangular coordinates below, select the correct conversion into polar coordinates. Don't use the polar conversion function on your calculator.

9. $E(-42, 5)$

A. $(42.3, -173.2°)$ B. $(42.3, 173.2°)$ C. $(-42.3, 173.2°)$

D. $(-42.3, -173.2°)$ E. $(-37, -6.8°)$

(a) 10. $G(-30, -12)$

A. $(32.3, -201.8°)$ B. $(-42, 21.8°)$ C. $(-32.3, 201.8°)$

D. $(-32.3, -201.8°)$ E. $(32.3, 201.8°)$

Solve each system of equations below.

11. $\begin{cases} 2x-4y=-5 \\ x-2y=3 \end{cases}$

12. $\begin{cases} 3x+4y=-1 \\ 6x-2y=3 \end{cases}$

(b) 13. $\begin{cases} y=x^2-3x+1 \\ x-y=2 \end{cases}$

(c) 14. $\begin{cases} 7x+5y+z=0 \\ -x+3y+2z=16 \\ x-6y-z=-18 \end{cases}$

Solve each system of equations below using a graphing calculator. Estimate any irrational answers to two decimal places.

15. $\begin{cases} y=x^3-4x^2+3x-1 \\ 2x+y=5 \end{cases}$

(d) 16. $\begin{cases} x^2+y^2=9 \\ y=x^4-2x^2+3x+3 \end{cases}$

Select the answer for each question below.

17. The graph of which of the following functions is symmetric with respect to the origin?

A. $g(x)=(x-5)^2$
B. $f(x)=e^x$
C. $F(x)=4\sin x$
D. $h(x)=(x+2)^3$
E. $G(x)=x^3-3$

18. What is the domain of $f(x)=\dfrac{1}{\sqrt{9-x^2}}$?

A. $x>-3$
B. $x<-3$ or $x>3$
C. $x\neq\pm3$
D. $\{x|x<3\}$
E. $\{x|-3<x<3\}$

Answer each question below. Estimate any irrational answers to two decimal places.

19. The sides of a triangle are 7, 8, and 9 centimeters. What is the measure in degrees of the angle opposite the 7 centimeters side?

20. If $7^{x+3}=8^x$, then $x=$?

(e) 21. $f(x)=\sin(\arctan x)$ and $g(x)=\tan(\arcsin x)$. If $0\le x\le\dfrac{\pi}{2}$, then $g\left(f\left(\dfrac{\pi}{9}\right)\right)=$?

Solve the word problem below.

(f) 22. For a fundraiser to go on a church mission trip, Penelope sold three different types of candy bars. Choconut Bars cost \$2, Almond-Bliss Bars cost \$1.50, and Fudge-Brownie Bars cost \$1. She ended up selling three times as many Fudge-Brownie bars than the other two combined. If Penelope sold 500 candy bars and made \$587.50, how many of each type of bar did she sell?

Lesson 66—Solving a System with Determinants

We've been reviewing how to solve systems of equations. And, as the examples have shown, the process can be pretty tedious. It would be nice if we had a shortcut for solving systems.

A Nice Shortcut

Well, the mathematicians agreed and came up with a pretty neat shortcut for solving certain kinds of systems. The quadratic formula is a shortcut for solving quadratic equations. Instead of having to factor or complete the square to solve a tough quadratic equation, we can just plug in the numbers on the right side of the formula, which is $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. The quadratic formula basically turns any quadratic equation into just an arithmetic problem, because we just have to do the arithmetic on the right side.

Remember, the quadratic formula is derived by solving the general equation $ax^2 + bx + c = 0$ for x in terms of the coefficients a, b, and c. Since any quadratic equation can fit into the form $ax^2 + bx + c = 0$, this is kind of like solving every possible quadratic equation at once. The shortcut for solving systems is derived in pretty much the same way. We start with a two-variable, two-equation system with letters in place of the coefficients.

$$\begin{cases} ax + by = c & \textbf{Equation 1} \\ dx + ey = f & \textbf{Equation 2} \end{cases}$$

It's possible to solve this system for x and y by elimination, even though the coefficients aren't known. We'll just end up with answers in terms of a, b, c, d, e, and f.

To eliminate the y's in both equations, we first must make their coefficients the same. We can do that by multiplying both sides of Equation 1 by e and both sides of Equation 2 by $-b$.

$$\begin{cases} e(ax + by) = ce & \textbf{Equation 1} \\ -b(dx + ey) = -b(f) & \textbf{Equation 2} \end{cases}$$

Simplifying gives us this.

$$\begin{cases} aex \;+\; bey = ce & \textbf{Equation 1} \\ -bdx + -bey = -bf & \textbf{Equation 2} \end{cases}$$

Now the y's are opposites, so we can add the sides of Equation 1 to the sides of Equation 2.

Added from Equation 1.

$$-bdx - bey + \overbrace{(aex + bey)} = -bf + ce$$

$$-bdx + -bey + (aex + bey) = -bf + ce$$

Now we can simplify. Of course, bey and $-bey$ cancel out.

$$-bdx + aex = -bf + ce$$

This looks complicated, but we can think of it as an equation with one unknown, x. The rest of the letters are just the constants. We can combine the two x-terms on the left side by reversing the distributing process. We just take an x outside parentheses.

$$x(-bd+ae)=-bf+ce$$

This gets the left side down to a single x-term, with a coefficient of $-bd+ae$. We can even put the coefficient in front of the x in the usual way.

$$(-bd+ae)x=-bf+ce$$

Now we finish solving by undoing the multiplication by x of $-bd+ae$. All we have to do is divide both sides of the equation by $-bd+ae$.

$$x=\frac{-bf+ce}{-bd+ae} \text{ or } x=\frac{ce-bf}{ae-bd}$$

Our answer for x, then, is $\frac{ce-bf}{ae-bd}$.

To find y, we need to substitute $\frac{ce-bf}{ae-bd}$ for x in either one of the original equations. We'll use Equation 1.

$$ax+by=c$$

$$a\left(\frac{ce-bf}{ae-bd}\right)+by=c$$ **substituting for x**

We multiply the fraction by a and then move the result to the right side.

$$\frac{ace-abf}{ae-bd}+by=c$$

$$by=c-\frac{ace-abf}{ae-bd}$$

Next, we subtract the fraction from c. The common denominator is $ae-bd$.

$$by=\frac{c(ae-bd)}{ae-bd}-\frac{ace-abf}{ae-bd}$$

Now we distribute the c in the top of the first fraction and subtract the tops.

$$by=\frac{ace-bcd}{ae-bd}-\frac{ace-abf}{ae-bd}$$

$$by=\frac{ace-bcd-(ace-abf)}{ae-bd}$$

Simplifying on top gives us this.

$$by = \frac{ace - bcd - ace + abf}{ae - bd}$$

$$by = \frac{abf - bcd}{ae - bd}$$

To finish solving for y, we need to get rid of the b on the left. The easiest way to do it is to multiply both sides by $\frac{1}{b}$.

$$\frac{1}{b} \cdot by = \frac{abf - bcd}{ae - bd} \cdot \frac{1}{b}$$

$$y = \frac{abf - bcd}{b(ae - bd)}$$

There's one last simplification step. We can factor out a b on top and cancel one pair of b's.

$$y = \frac{b(af - cd)}{b(ae - bd)}$$

$$y = \frac{af - cd}{ae - bd}$$

So our answers for the system are $x = \frac{ce - bf}{ae - bd}$ and $y = \frac{af - cd}{ae - bd}$.

Make sure you understand what these answers mean. For a specific two-variable system, with actual numbers in for the letters a through f, we could find the system's solutions for x and y just by substituting the value of the letters into the fractions $\frac{ce - bf}{ae - bd}$ and $\frac{af - cd}{ae - bd}$. It's just like substituting the values or a, b, and c in the quadratic formula, $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. To show you how it works, let's take one of the systems we solved a couple of lessons ago.

Now let's look at an actual system. This one is from a couple of lessons ago.

$$\begin{cases} 2x + 3y = -4 & \textbf{Equation 1} \\ 3x - 5y = -25 & \textbf{Equation 2} \end{cases}$$

Instead of having to go through the long process of having to solve this system with the elimination method, all we have to do is put the values for a, b, c, d, e, and f into the formulas $x = \dfrac{ce - bf}{ae - bd}$ and $y = \dfrac{af - cd}{ae - bd}$. For our system, $a = 2$, $b = 3$, $c = -4$, $d = 3$, $e = -5$, and $f = -25$. Plugging those numbers in gives us this.

$$x = \frac{(-4)(-5) - (3)(-25)}{(2)(-5) - (3)(3)} \qquad\qquad y = \frac{(2)(-25) - (-4)(3)}{(2)(-5) - (3)(3)}$$

$$x = -5 \qquad\qquad y = 2$$

These are the exact same answers we got before.

Using Determinants

There's a simple and important way to remember the formulas $x = \dfrac{ce - bf}{ae - bd}$ and $y = \dfrac{af - cd}{ae - bd}$. Notice that the denominators are the same, and that they contain the letters a, b, d, and e, which represent the coefficients for x and y in the equations $ax + by = c$ and $dx + ey = f$. To remember the formula for the denominators, what we do is write these coefficients down in the shape of a rectangle, and list them in the same order that they appear in the equations.

$$\begin{vmatrix} a & b \\ d & e \end{vmatrix}$$

Now to find the expression for the denominators, all we do is cross multiply and subtract the products.

$$\begin{vmatrix} a & b \\ d & e \end{vmatrix} = ae - bd$$

This four-number symbol with vertical bars is called a **determinant**. The numbers inside are called elements. And the value of the determinant is calculated by cross multiplying and subtracting the products. We'll call the determinant for the denominator D, so we have $D = \begin{vmatrix} a & b \\ d & e \end{vmatrix} = ae - bd$.

The numerators of our x and y formulas can also be written as determinants. The x numerator is $ce - bf$. This doesn't include the coefficients of the x-terms in the equations $ax + by = c$ and $dx + ey = f$. To set up the determinant for the x numerator, then, all you have to do is start with $D = \begin{vmatrix} a & b \\ d & e \end{vmatrix}$ and replace the x-coefficients with the letters on the right side of both equations. In other words, replace a and d with c and f.

$$\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$$

This gives us the determinant $\begin{vmatrix} c & b \\ f & \text{e} \end{vmatrix}$, which we'll call N_x, since it's the determinant for the *x*-value of the numerator. Using the same rule to calculate the value of this determinant—cross multiplying and subtracting the products—gives us the correct expression for the numerator of the *x*-value.

$$N_x = \begin{vmatrix} c & b \\ f & e \end{vmatrix} = ce - bf$$

To find the numerator for the *y*-value, we just start with $D = \begin{vmatrix} a & b \\ d & e \end{vmatrix}$ again. Only this time, we replace the *y*-coefficients with the letters in the right side of the equations $ax + by = c$ and $dx + ey = f$. The *y*-coefficients are *b* and *e*. So we take out *b* and *e* and put in *c* and *f*.

$$\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$$

That gives us a new determinant, $\begin{vmatrix} a & c \\ d & f \end{vmatrix}$, which we'll call N_y, since it's the determinant for the *y*-value of the numerator. Using the same rule to calculate the value of this determinant—cross multiplying and subtracting the products—gives us the correct expression for the numerator of the *y*-value.

$$N_y = \begin{vmatrix} a & c \\ d & f \end{vmatrix} = af - cd$$

So the formulas for the solutions to any two-variable, two-equation system $\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$ can be written using determinants.

$$x = \frac{\begin{vmatrix} c & b \\ f & e \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}} \text{ and } y = \frac{\begin{vmatrix} a & c \\ d & f \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}} \qquad \text{or} \qquad x = \frac{N_x}{D} \text{ and } y = \frac{N_y}{D}$$

To solve any system of the form $\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$, all you have to do is remember that the denominators equal a determinant with the coefficients of *x* and *y* (in the order they appear in the equations), and the numerators are determinants that replace the coefficients of the particular variable (*x* or *y*) with the numbers on the right side of the equals sign. Then you just set up those determinants and calculate their value by cross multiplying and subtracting. This method of solving a system is also called Cramer's Rule, after a famous mathematician. To give you one last quick example, let's solve the system below with determinants.

$$\begin{cases} 6x - 7y = 47 \\ 2x + 5y = -21 \end{cases}$$

We just put the numbers for *a* through *f* into the determinants in their proper places.

$$x=\frac{\begin{vmatrix}47 & -7\\ -21 & 5\end{vmatrix}}{\begin{vmatrix}6 & -7\\ 2 & 5\end{vmatrix}} \text{ and } y=\frac{\begin{vmatrix}6 & 47\\ 2 & -21\end{vmatrix}}{\begin{vmatrix}6 & -7\\ 2 & 5\end{vmatrix}}$$

Now cross multiplying and subtracting the products for each determinant gives us the answers.

$$x=\frac{47(5)-(-7)(-21)}{6(5)-(-7)(2)} \text{ and } y=\frac{6(-21)-47(2)}{6(5)-(-7)(2)}$$

$$x=\frac{88}{44}=2 \text{ and } y=\frac{-220}{44}=-5$$

The solutions to the system are $x=2$ and $y=-5$.

Practice 66

Solve each system of equations below.

a. $\begin{cases} y=4x^3+x^2 \\ y=9x^2 \end{cases}$

b. $\begin{cases} 2x+y+4z=13 \\ 3x-y-2z=-1 \\ 4x+2y+z=19 \end{cases}$

c. Solve the system using determinants: $\begin{cases} -5x+11y=13 \\ 2x+8y=32 \end{cases}$

d. Select the expression below that is identical to $\cos^4\alpha-\sin^4\alpha$.

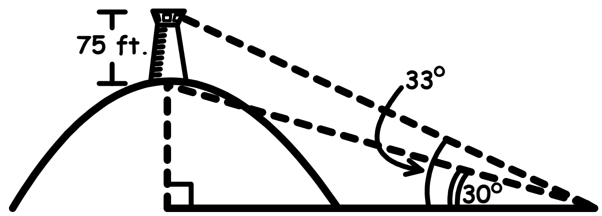
A. $2\cos^2\alpha$ B. 1 C. $\cos 2\alpha$
D. $\cos^4\alpha$ E. $\sin 2\alpha$

e. Which of the following quadratic equations has roots $7+i$ and $7-i$?

A. $x^2+14x+53=0$ B. $x^2-14x+50=0$ C. $x^2+14x+50=0$
D. $x^2+14x-50=0$ E. $x^2+14x-53=0$

f. Solve the word problem below.

An observation tower that is 75 feet tall sits on top of a hill. From a point at the base of the hill, the angles of elevation to the top and bottom of the antenna are $33°$ and $30°$, respectively. To the nearest whole number of feet, how high is the hill?



Problem Set 66

Tell whether each sentence below is True or False.

1. The value of a determinant can be found by cross multiplying and adding the products.

2. Using determinants to solve systems of equations is called Cramer's Rule.

If $\mathbf{u} = -7\mathbf{i} + 25\mathbf{j}$ and $\mathbf{v} = 14\mathbf{i} - 9\mathbf{j}$, calculate the value of each expression below. Estimate any irrational answers to two decimal places.

3. $3\mathbf{u} - 2\mathbf{v}$

4. $\|\mathbf{u} + \mathbf{v}\|$

For each pair of coordinates below, use the polar conversion function of your calculator to select the correct conversion.

5. Convert $H(31, 152°)$ to rectangular coordinate

A. $(-27.4, 14.6)$ B. $(-27.4, -14.6)$ C. $(27.4, 14.6)$
D. $(14.6, -27.4)$ E. $(27.4, -14.6)$

6. Convert $I(16, -9)$ to polar coordinate

A. $(-18.4, -29.4°)$ B. $(18.4, 29.4°)$ C. $(18.4, -29.4°)$
D. $(-18.4, 29.4°)$ E. $(-29.4, 18.4°)$

Solve each system of equations below.

7. $\begin{cases} y = x^2 + 3x - 7 \\ 2x - y = 1 \end{cases}$

(a) 8. $\begin{cases} y = 3x^3 + 2x^2 \\ y = 8x^2 \end{cases}$

(b) 9. $\begin{cases} 3x-5y+z=9 \\ x-3y-2z=-8 \\ 5x-6y+3z=15 \end{cases}$

Solve each system of equations below using a graphing calculator. Estimate any irrational answers to one decimal place.

10. $\begin{cases} y=x^3-4x^2+3x-1 \\ 2x+y=5 \end{cases}$

11. $\begin{cases} x^2+y^2=15 \\ y=x^3-5x^2+2x+2 \end{cases}$

Solve the systems below using determinants.

12. $\begin{cases} x+y=-2 \\ 3x-y=10 \end{cases}$

(c) 13. $\begin{cases} -3x+10y=5 \\ 2x+7y=24 \end{cases}$

Answer each question below.

14. Select the expression below that is identical to $\dfrac{\csc\alpha}{\cot^2\alpha}$.

A. $\sin\alpha$
B. $\sec\alpha$
C. $\csc\alpha$
D. $\tan\alpha\sec\alpha$
E. $\cot\alpha\csc\alpha$

(d) 15. Select the expression below that is identical to $\cos^2\alpha-\sin^2\alpha$.

A. $2\cos\alpha$
B. 1
C. $\sin 2\alpha$
D. $\cos^2\alpha$
E. $\cos 2\alpha$

16. If $x=2\sec^2\theta$ and $y=1-\sin^2\theta$, what is y in terms of x?

Solve each trig equation below for $0\le x\le\pi$. (All angles are in radians.)

17. $2\cos^2 x-1=0$

18. $2\sin^2 x+3\cos x-3=0$

Select the answer for each question below.

19. If a point has polar coordinates $(5,\pi)$, then what are its rectangular coordinates?

A. $(-5,-5)$
B. $(-5,0)$
C. $(0,-5)$
D. $(5,0)$
E. $(0,5)$

(e) 20. Which of the following quadratic equations has roots $4+i$ and $4-i$?

A. $x^2+8x-17=0$ B. $x^2+8x-19=0$ C. $x^2+8x+17=0$
D. $x^2+8x+19=0$ E. $x^2-8x+17=0$

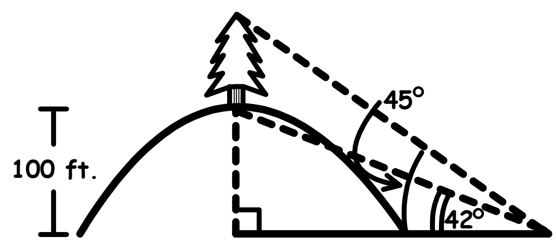
21. Which of the following lines are asymptotes of the graph of $f(x)=\dfrac{2(x^2-4)}{x^2-9}$?

I. $x=\pm 3$ II. $x=2$ III. $y=2$

A. I only B. I and II only C. II only
D. I and III only E. I, II, and III

Solve the word problem below.

(f) 22. A tree sits on top of a hill that is 100 feet tall. From a point at the base of the hill, the angles of elevation to the top and bottom of the tree are $45°$ and $42°$, respectively. To the nearest whole number of feet, how tall is the tree?



Lesson 67—More on Determinants

In the last lesson, we learned a shortcut for solving any two-variable, two-equation system. The shortcut is kind of like the quadratic formula, and the easiest way to remember it is to use determinants. For instance, in any system of the form $\begin{cases} ax+by=c \\ dx+ey=f \end{cases}$, the solutions have to be $x=\dfrac{ce-bf}{ae-bd}$ and $y=\dfrac{af-cd}{ae-bd}$. But to help remember these solutions, we write them using the determinants below.

$$x=\frac{\begin{vmatrix} c & b \\ f & \mathrm{e} \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}} \qquad y=\frac{\begin{vmatrix} a & c \\ d & f \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}}$$

To find the numerators and denominators, all we have to do is calculate the value of each determinant by cross multiplying and subtracting the products. That gives $x=\dfrac{ce-bf}{ae-bd}$ and $y=\dfrac{af-cd}{ae-bd}$.

3 x 3 Determinants

A determinant with four numbers inside the vertical bars (two rows and two columns) is actually called a two-by-two (written 2×2) determinant.

$$\begin{vmatrix} 47 & -7 \\ -21 & 5 \end{vmatrix}$$

2 x 2 determinant

This kind of determinant is used to write the formula for solving any linear system with two variables and two equations. But what about three-variable, three-equation systems? As it turns out, we can write a formula for those as well, but it requires that we use 3×3 determinants, which are determinants that have 9 numbers inside the vertical bars: three rows and three columns. Here's an example.

$$\begin{vmatrix} 47 & -7 & 2 \\ -21 & 5 & 3 \\ 9 & 4 & -1 \end{vmatrix}$$

3 x 3 determinant

We can't just cross multiply and subtract the products to calculate the value of a 3×3 determinant; there are too many numbers for that. The method for calculating a 3×3 determinant is a little more complicated. We start by writing down the number in the upper left, which is 47. Then we cross out all the other numbers in the same row and in the same column.

$$47\begin{vmatrix} 47 & \cancel{-7} & \cancel{2} \\ \cancel{-21} & 5 & 3 \\ \cancel{9} & 4 & -1 \end{vmatrix}$$

What's left is a 2×2 determinant, $\begin{vmatrix} 5 & 3 \\ 4 & -1 \end{vmatrix}$. So we write 47 in front of this new determinant. Actually, it's called a minor determinant. And what we're doing is multiplying 47 by the minor determinant.[2] We'll show you how the multiplication works in a second.

$$47\begin{vmatrix} 5 & 3 \\ 4 & -1 \end{vmatrix}$$

The next step is to write down the number to the immediate right of 47, which is -7. Then we cross out all the other numbers in the same row and in the same column again.

$$-7\begin{vmatrix} \cancel{47} & -7 & \cancel{2} \\ -21 & \cancel{5} & 3 \\ 9 & \cancel{4} & -1 \end{vmatrix}$$

This leaves another minor determinant, $\begin{vmatrix} -21 & 3 \\ 9 & -1 \end{vmatrix}$. So we multiply this minor determinant by -7, and subtract it from $47\begin{vmatrix} 5 & 3 \\ 4 & -1 \end{vmatrix}$.

$$47\begin{vmatrix} 5 & 3 \\ 4 & -1 \end{vmatrix} - (-7)\begin{vmatrix} -21 & 3 \\ 9 & -1 \end{vmatrix}$$

Continuing the process, we move across the top to the next number, 2. Crossing out the same row and same column leaves the minor determinant $\begin{vmatrix} -21 & 5 \\ 9 & 4 \end{vmatrix}$. We multiply this determinant by 2 and add the result to the others.

$$47\begin{vmatrix} 5 & 3 \\ 4 & -1 \end{vmatrix} - (-7)\begin{vmatrix} -21 & 3 \\ 9 & -1 \end{vmatrix} + 2\begin{vmatrix} -21 & 5 \\ 9 & 4 \end{vmatrix}$$

Now we calculate each of the minor determinants, by cross multiplying and subtracting the products.

$$47[5(-1)-3(4)]-(-7)[-21(-1)-3(9)]+2[-21(4)-5(9)]$$

Simplifying gives us this.

$$47(-5-12)+7(21-27)+2(-84-45)$$

$$47(-17)+7(-6)+2(-129)$$

$$-799+-42+-258$$

$$-1{,}099$$

The value of the 3×3 determinant is $-1{,}099$.

[2] Actually, this same method will work when calculating a 2 by 2 determinant. After crossing out the row and column and multiplying the chosen number by what's left, you still end up cross multiplying.

3 x 3 Determinants and Solving a System

Since 3×3 determinants have enough room for 9 numbers, they can be used to solve three-variable, three equation systems. That's enough room for a coefficient for *x*, *y*, and *z* for each of the three equations. Let's solve the three-equation system from a couple of lessons ago using determinants.

$$\begin{cases} 5x-4y+3z=15 & \textbf{Equation 1} \\ 6x+2y+9z=13 & \textbf{Equation 2} \\ 7x+6y-6z=6 & \textbf{Equation 3} \end{cases}$$

The process is basically the same as the one we use for two equation systems. The first step is to create a determinant containing the coefficients for *x*, *y*, and *z* in each equation—in the same order they appear in the equations. Our system is in the form $\begin{cases} ax+by+cz=p \\ dx+ey+fz=q \\ gx+hy-kz=s \end{cases}$, so the first determinant should have the coefficients in the following order.

$$D=\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix}$$

This is the determinant for the denominator of all three solutions of the system. Now we put the numbers in.

$$D=\begin{vmatrix} 5 & -4 & 3 \\ 6 & 2 & 9 \\ 7 & 6 & -6 \end{vmatrix}$$

Next, we calculate the value of the determinant, following the rules explained above.

$$\begin{vmatrix} 5 & \cancel{-4} & \cancel{3} \\ \cancel{6} & 2 & 9 \\ \cancel{7} & 6 & -6 \end{vmatrix} \qquad \begin{vmatrix} \cancel{5} & -4 & \cancel{3} \\ 6 & \cancel{2} & 9 \\ 7 & \cancel{6} & -6 \end{vmatrix} \qquad \begin{vmatrix} \cancel{5} & \cancel{-4} & 3 \\ 6 & 2 & \cancel{9} \\ 7 & 6 & \cancel{-6} \end{vmatrix}$$

Now we multiply each number by its minor determinant. The second term is subtracted and the third term is added.

$$D \quad = \quad 5\begin{vmatrix} 2 & 9 \\ 6 & -6 \end{vmatrix} \quad - \quad -4\begin{vmatrix} 6 & 9 \\ 7 & -6 \end{vmatrix} \quad + \quad 3\begin{vmatrix} 6 & 2 \\ 7 & 6 \end{vmatrix}$$

Next, we cross multiply and subtract the products in each minor determinant and simplify the entire expression.

$$D=5[2(-6)-9(6)]--4[6(-6)-9(7)]+3[6(6)-2(7)]$$

$$D=5(-66)+4(-99)+3(22)$$

$$D=-330-396+66$$

$$D=-660$$

The denominator for *x*, *y*, and *z* equal -660.

What about the numerators? The process is basically the same one we use for two-variable, two-equation systems. Starting with the determinant for the denominator, to find the numerator for *x*, we replace the coefficients of x with the numbers on the right side of the equals sign in the three equations of the system. Here are the equations for the system and the determinant for the denominator again

$$D = \begin{vmatrix} 5 & -4 & 3 \\ 6 & 2 & 9 \\ 7 & 6 & -6 \end{vmatrix} \qquad \begin{cases} 5x - 4y + 3z = 15 \\ 6x + 2y + 9z = 13 \\ 7x + 6y - 6z = 6 \end{cases}$$

The x-coefficients are in the first column (5, 6, and 7). We need to replace those with 15, 13, and 6.

$$N_x = \begin{vmatrix} 15 & -4 & 3 \\ 13 & 2 & 9 \\ 6 & 6 & -6 \end{vmatrix}$$

Now we calculate this determinant by the usual method. We multiply 15, -4, and 3 by three minor determinants.

$$\begin{vmatrix} 15 & -4 & 3 \\ 13 & 2 & 9 \\ 6 & 6 & -6 \end{vmatrix} \qquad \begin{vmatrix} 15 & -4 & 3 \\ 13 & 2 & 9 \\ 6 & 6 & -6 \end{vmatrix} \qquad \begin{vmatrix} 15 & -4 & 3 \\ 13 & 2 & 9 \\ 6 & 6 & -6 \end{vmatrix}$$

$$N_x = 15\begin{vmatrix} 2 & 9 \\ 6 & -6 \end{vmatrix} - -4\begin{vmatrix} 13 & 9 \\ 6 & -6 \end{vmatrix} + 3\begin{vmatrix} 13 & 2 \\ 6 & 6 \end{vmatrix}$$

Next, we cross multiply and subtract the products for each minor determinant.

$$N_x = 15[2(-6) - 9(6)] - -4[13(-6) - 9(6)] + 3[13(6) - 2(6)]$$

$$N_x = 15(-66) + 4(-132) + 3(66)$$

$$N_x = -1,320$$

The last step for finding x is to put together the fraction with $N_x = -1,320$ on top and $D = -660$ on bottom.

$$x = \frac{N_x}{D} = \frac{-1,320}{-660} = 2$$

So the solution for x of our system is 2. This is the same answer we got a couple of lessons ago.

We need to go through the same steps to find the answers for y and z. Only since the denominator for these will be the same as x, we can skip that step. We already know that $D = -660$. The find the determinant for the numerator of y, we start with D and then replace the y-coefficients with the numbers on the right side of our three equations.

$$D = \begin{vmatrix} 5 & -4 & 3 \\ 6 & 2 & 9 \\ 7 & 6 & -6 \end{vmatrix} \qquad \begin{cases} 5x - 4y + 3z = 15 \\ 6x + 2y + 9z = 13 \\ 7x + 6y - 6z = 6 \end{cases}$$

$$N_y = \begin{vmatrix} 5 & 15 & 3 \\ 6 & 13 & 9 \\ 7 & 6 & -6 \end{vmatrix}$$

Now we calculate this 3×3 determinant according to our method.

$$\begin{vmatrix} 5 & 15 & 3 \\ 6 & 13 & 9 \\ 7 & 6 & -6 \end{vmatrix} \qquad \begin{vmatrix} 5 & 15 & 3 \\ 6 & 13 & 9 \\ 7 & 6 & -6 \end{vmatrix} \qquad \begin{vmatrix} 5 & 15 & 3 \\ 6 & 13 & 9 \\ 7 & 6 & -6 \end{vmatrix}$$

$$N_y \quad = \quad 5\begin{vmatrix} 13 & 9 \\ 6 & -6 \end{vmatrix} \quad - \quad 15\begin{vmatrix} 6 & 9 \\ 7 & -6 \end{vmatrix} \quad + \quad 3\begin{vmatrix} 6 & 13 \\ 7 & 6 \end{vmatrix}$$

$$N_y = 5[13(-6)-9(6)]-15[6(-6)-9(7)]+3[6(6)-13(7)]$$

$$N_y = 5(-132)-15(-99)+3(-55)$$

$$N_y = 660$$

So $y = \dfrac{N_y}{D} = \dfrac{660}{-660} = -1$. Finally, we go through the same process for the numerator of *z*. We start with *D* and then replace the *z*-coefficients with the numbers on the right side of our three equations.

$$D = \begin{vmatrix} 5 & -4 & 3 \\ 6 & 2 & 9 \\ 7 & 6 & -6 \end{vmatrix} \qquad\qquad \begin{cases} 5x-4y+3z=15 \\ 6x+2y+9z=13 \\ 7x+6y-6z=6 \end{cases}$$

$$N_z = \begin{vmatrix} 5 & -4 & 15 \\ 6 & 2 & 13 \\ 7 & 6 & 6 \end{vmatrix}$$

Next, we calculate this 3×3 determinant

$$\begin{vmatrix} 5 & -4 & 15 \\ 6 & 2 & 13 \\ 7 & 6 & 6 \end{vmatrix} \qquad \begin{vmatrix} 5 & -4 & 15 \\ 6 & 2 & 13 \\ 7 & 6 & 6 \end{vmatrix} \qquad \begin{vmatrix} 5 & -4 & 15 \\ 6 & 2 & 13 \\ 7 & 6 & 6 \end{vmatrix}$$

$$N_z \quad = \quad 5\begin{vmatrix} 2 & 13 \\ 6 & 6 \end{vmatrix} \quad - \quad -4\begin{vmatrix} 6 & 13 \\ 7 & 6 \end{vmatrix} \quad + \quad 15\begin{vmatrix} 6 & 2 \\ 7 & 6 \end{vmatrix}$$

$$N_z = 5[2(6)-13(6)]--4[6(6)-13(7)]+15[6(6)-2(7)]$$

$$N_z = 5(-66)+4(-55)+15(22)$$

$$N_z = -220$$

Putting this numerator on top of the denominator, $D=-660$, gives us $y = \dfrac{N_z}{D} = \dfrac{-220}{-660} = \dfrac{1}{3}$. The final answers to the system, then, are $x=2$, $y=-1$, and $z=\dfrac{1}{3}$, which are the same answers we got before.

Practice 67

a. Solve the system of equations: $\begin{cases} x^2 + y^2 = 29 \\ 5y = 2x \end{cases}$

b. Find the value of the determinant: $\begin{vmatrix} -4 & 2 & 5 \\ 2 & 0 & -3 \\ 1 & -2 & 3 \end{vmatrix}$

c. Solve the system of equations $\begin{cases} -3x + 5y - 4z = -5 \\ 4x - 2y + 3z = 2 \\ 7x + y + 5z = 7 \end{cases}$ using determinants.

d. Use your knowledge of trig identities (not a calculator) to find each trig value below.
If $\sec\theta = \dfrac{7}{4}$, find $\cos\theta + \cos\theta\tan^2\theta$

e. The amount of potential energy stored in a compressed string varies jointly (directly) with one-half of the square of the compression of the spring (x). If the 80 joules of energy are stored when the spring is compressed by 0.5 centimeters, how many joules of energy would be stored by the same spring compressed by 2 centimeters?

Problem Set 67

Tell whether each sentence below is True or False.

1. Determinants can be used to solve three-variable, three-equation systems.

2. The method for evaluating a 3×3 determinant is the same as evaluating a 2×2 determinant.

Answer each question below. Estimate any irrational answers to one decimal place.

3. Find the magnitude and direction angle of $\mathbf{m} = 57\mathbf{i} + 14\mathbf{j}$.

4. Find the unit vector that's in the same direction as $\mathbf{u} = 24\mathbf{i} - 7\mathbf{j}$.

Solve each system of equations below.

5. $\begin{cases} 2x - 4y = -3 \\ -6x + 12y = 9 \end{cases}$

6. $\begin{cases} y = x^2 - x - 6 \\ y = 2x - 2 \end{cases}$

(a) 7. $\begin{cases} x^2 + y^2 = 25 \\ 4y = 3x \end{cases}$

Find the value of each determinant below.

8. $\begin{vmatrix} 9 & 4 \\ 3 & -2 \end{vmatrix}$

(b) 9. $\begin{vmatrix} -2 & 1 & 3 \\ 4 & 0 & -1 \\ 1 & -3 & 5 \end{vmatrix}$

Solve each system of equations below using determinants.

10. $\begin{cases} 2x+4y=6 \\ 5x-3y=41 \end{cases}$

11. $\begin{cases} 3x+2y=2 \\ -9x+4y=-1 \end{cases}$

(c) 12. $\begin{cases} -4x+7y-3z=6 \\ 3x-2y+4z=4 \\ 5x+y+5z=6 \end{cases}$

Use your knowledge of trig identities (not a calculator) to find each trig value below.

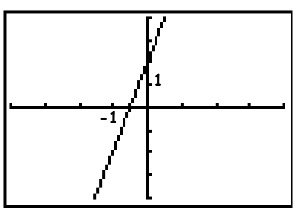
13. If $\cos(-\theta)=\frac{1}{3}$, find $\sin(\frac{\pi}{2}-\theta)$

(d) 14. If $\csc\alpha=\frac{5}{2}$, find $\sin\alpha+\sin\alpha\cot^2\alpha$

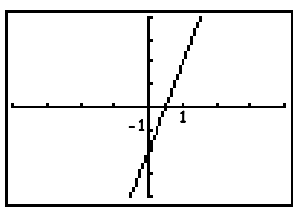
Select the correct graph of the pair of parametric equations and polar equation below.

15. $x=\frac{1}{2}t-1$, $y=2t-6$

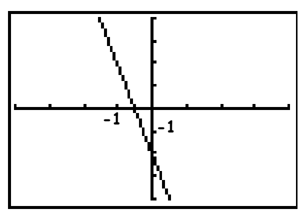
A.



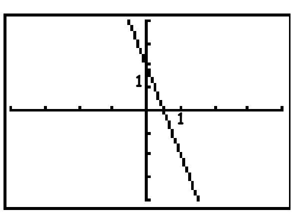
B.



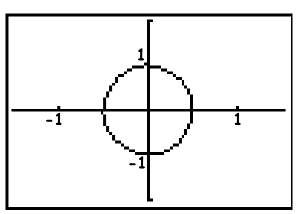
C.



D.

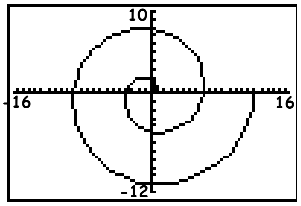


E.

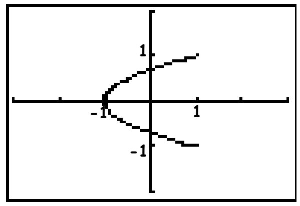


(e) 16. $r = 3\cos 4\theta$ (The angle is in radians)

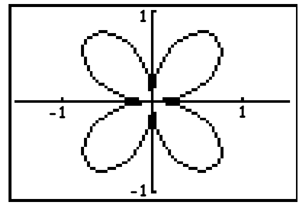
A.



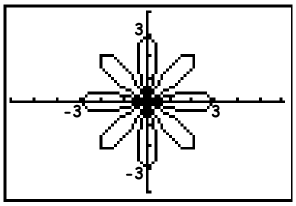
B.



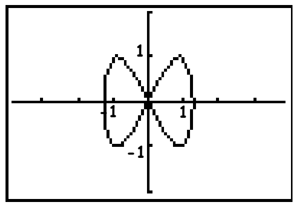
C.



D.

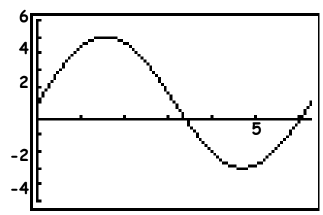


E.



Select the answer for each question below.

17. The figure below shows one period of the graph of $y = 4\sin x + 1$ for $0 \le x < 2\pi$. What are the coordinates of the point where the minimum value of the function occurs on this interval?



A. $\left(\dfrac{\pi}{2}, 5\right)$

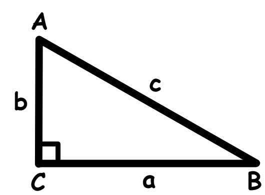
B. $(\pi, -3)$

C. $\left(\dfrac{3\pi}{2}, -3\right)$

D. $\left(\dfrac{5\pi}{4}, -4\right)$

E. $\left(\dfrac{3\pi}{2}, -4\right)$

18. In ΔABC, $\dfrac{\cot A \cos B}{\csc B} = ?$



A. $\dfrac{a^3}{bc^2}$

B. $\dfrac{a^2}{c^2}$

C. 1

D. $\dfrac{b^2}{c^2}$

E. $\dfrac{a^2 b}{c^3}$

19. What is the range of $f(x) = -8\sin\dfrac{x}{5}$?

A. All real numbers greater than or equal to 0 and less than or equal to 5

B. All real numbers greater than or equal to -8 and less than or equal to 8.

C. All real numbers greater than or equal to $-\dfrac{1}{5}$ and less than or equal to $\dfrac{1}{5}$.

D. All real numbers greater than or equal to 0 and less than or equal to $\dfrac{1}{5}$.

E. All real numbers greater than or equal to -8 and less than or equal to 0.

Answer each question below. Estimate any irrational answers to two decimal places.

20. If $f(x) = e^{3x}$ and $g(x) = x^{-4}$, then $f(g(2)) = ?$

21. If $f(x) = x + 5$ and $g(x) = 7 - x^2$. What is the maximum value of $g(f(x))$?

Solve the word problem below.

(f) 22. According to Boyle's Law, the pressure that the gas at a constant temperature exerts on a container is indirectly proportional (varies inversely) with the volume of the container. If a gas exerts 1.2 atmospheres of pressure on a container with a volume of 2 liters, how much pressure would the gas exert on a container with a volume of 4 liters?

Lesson 68—Solving Systems with Matrices

We've been learning how to solve systems using determinants. The nice thing about determinants is that they allow us to solve any linear system with just arithmetic. No algebra is needed. The drawback of determinants is that as the number of variables and equations in a system goes up (beyond three), the amount of arithmetic required increases hugely. Since complicated real world problems can sometimes involve systems with thousands of variables, we need a better shortcut method than determinants.

Making a Triangle

But before we can explain a better shortcut, we need to talk a little about systems with lots of variables. A three-variable, three-equation system is easy to solve once it's been worked down into a form like this.

$$\begin{cases} 3x + y + 2z = 1 \\ 2y + 3z = 8 \\ -4z = 16 \end{cases}$$

Since the bottom equation just has one unknown, all we have to do is solve it for z and then substitute the result in for z in the middle equation to get an answer for y.

$$-4z = 16 \qquad\qquad 2y + 3z = 8$$

$$z = -4 \qquad\qquad 2y + 3(-4) = 8$$

$$2y = 20$$

$$y = 10$$

And once we know that $z = -4$ and $y = 10$, we can substitute those numbers into the top equation to solve for x.

$$3x + y + 2y = 1$$

$$3x + 10 + 2(-4) = 1$$

$$3x = -1$$

$$x = -\frac{1}{3}$$

So the solutions are $x = -\frac{1}{3}$, $y = 10$, and $z = -4$.

An easy system like this is said to be in "triangular" form, because the equations are shaped a little like a triangle.

$$\begin{cases} 3x + y + 2z = 1 \\ 2y + 3z = 8 \\ -4z = 16 \end{cases}$$

See, since the equations get shorter as you go from top to bottom, the system is a little like an upside down triangle. The second shortcut for solving systems that we're about to learn involves turning a system into triangular form.

From System to Matrix

In order to make the steps as fast and simple as possible, this shortcut uses something called a matrix. A matrix is basically just a group of numbers laid out in rows and columns in the shape of a rectangle. More technically, we could say that a matrix is a "rectangular array" of numbers. Here's an example of a matrix.

$$\begin{bmatrix} 5 & 0 & -7 \\ 8 & 1 & 14 \end{bmatrix}$$

The numbers are in rows and columns. It's similar to a determinant written with vertical bars. Only with a matrix, instead of plain vertical bars, there are brackets on either side. And notice that the matrix has more columns than rows. A matrix can have a different number of rows than columns; all sorts of combinations are possible. That's different from determinants, which are square.

Here's how we represent a three-variable, three-equation system with a matrix. We'll set up a matrix for the system below.

$$\begin{cases} 2x+3y-z=-1 \\ -x+5y+3z=-10 \\ 3x-y-6z=5 \end{cases}$$

Inside the matrix, we put the coefficients of x, y, and z, and the numbers on the right side of the equals sign for each equation. And these numbers should be in the same order as they appear in the system.

$$\left[\begin{array}{rrr|r} 2 & 3 & -1 & -1 \\ -1 & 5 & 3 & -10 \\ 3 & -1 & -6 & 5 \end{array}\right]$$

We also draw a vertical line inside the matrix that separates the coefficients from the numbers on the right. The vertical line serves as a simplified equals signs for the three equations. This matrix is a simplified way to write the system. It includes only the essential information: the coefficients and the numbers on the right. The x's, y's, z's, and equals signs are left out.

We can do the same kinds of things to the matrix that we can to a system. Since it's legal to multiply all the terms of an equation in a system by the same number (that's the same as multiplying both sides by the same number), we can multiply all of the numbers in any row of our matrix by the same number. And since it's legal to add or subtract the like terms of one equation in a system to the like terms of one of the other equations, we're allowed to add or subtract all of the numbers in one row of the matrix to the corresponding numbers of any other row.

When we multiply all of the terms of a row by some number, we just replace the old numbers with the new ones. For instance, if we multiplied the top row by 3, the matrix would change from the original form to the form on the right below.

$$\left[\begin{array}{rrr|r} \mathbf{2} & \mathbf{3} & \mathbf{-1} & \mathbf{-1} \\ -1 & 5 & 3 & -10 \\ 3 & -1 & -6 & 5 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} \mathbf{6} & \mathbf{9} & \mathbf{-3} & \mathbf{-3} \\ -1 & 5 & 3 & -10 \\ 3 & -1 & -6 & 5 \end{array}\right]$$

What we really did with this step was to multiply both sides of the first equation by 3 to create a new equation.

$$\begin{cases} 3(2x+3y-z)=3(-1) \\ -x+5y+3z=-10 \\ 3x-y-6z=5 \end{cases} \longrightarrow \begin{cases} 6x+9y-3z=-3 \\ -x+5y+3z=-10 \\ 3x-y-6z=5 \end{cases}$$

The new system is equivalent to the original. It has the exact same solutions as the original system. But notice that the new matrix above has the same coefficients and numbers as the new system. So when we multiply all the terms of a row in a matrix by the same number, the result is an equivalent matrix.

When adding (or subtracting) the terms of one row of a matrix to those of another, we also just replace the old row with the new. For example, we add row 2 to row 3 by just replacing row 3 with the sum of the two rows.

$$\left[\begin{array}{rrr|r} 2 & 3 & -1 & -1 \\ \mathbf{-1} & \mathbf{5} & \mathbf{3} & \mathbf{-10} \\ \mathbf{3} & \mathbf{-1} & \mathbf{-6} & \mathbf{5} \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 2 & 3 & -1 & -1 \\ -1 & 5 & 3 & -10 \\ \mathbf{2} & \mathbf{4} & \mathbf{-3} & \mathbf{-5} \end{array}\right]$$

This step is the same as adding the sides of Equation 2 to the sides of Equation 3 and creating a new equation. Replacing Equation 3 with that new equation creates a system that's equivalent to the original.

$$\begin{cases} 2x + 3y - z = -1 \\ \mathbf{-1}x + \mathbf{5}y + \mathbf{3}z = \mathbf{-10} \\ \mathbf{3}x - \mathbf{1}y - \mathbf{6}z = \mathbf{5} \end{cases} \longrightarrow \begin{cases} 2x + 3y - z = -1 \\ -x + 5y + 3z = -10 \\ \mathbf{2}x + \mathbf{4}y - \mathbf{3}z = \mathbf{-5} \end{cases}$$

Here are the rules for what can be done to a matrix.

Table 68.1
Arithmetic that can be done to matrices.

1.	Multiply each number in a row by the same number and replace the old row with the new one.
2.	Add (or subtract) the numbers of one row to (from) the corresponding numbers of another row and replace the old row with the new one.
3.	Interchange any two rows.

An Actual Example

Now let's go through an actual example. We'll solve the system below using matrices. The matrix form is on the right.

$$\begin{cases} 2x+3y-z=-1 \\ -x+5y+3z=-10 \\ 3x-y-6z=5 \end{cases} \qquad \left[\begin{array}{rrr|r} 2 & 3 & -1 & -1 \\ -1 & 5 & 3 & -10 \\ 3 & -1 & -6 & 5 \end{array}\right]$$

The basic strategy is to change the rows of the matrix so that the system ends up in triangular form. Remember, in that form the system is really easy to solve. Here's what a matrix looks like when it's in triangular form.

$$\left[\begin{array}{rrr|r} 2 & 3 & -1 & -1 \\ 0 & 5 & 3 & -10 \\ 0 & 0 & -6 & 5 \end{array}\right]$$

Notice the bottom row now has 0s in the two positions on the left. And the middle row has one zero on the far left. If you compare this to the original three equations, you'll see why this qualifies as triangular form. The bottom row means $-6z = 5$, which is a simple equation with one unknown that can be solved easily. The middle row means $5y + 3z = -10$ is a two-variable equation that can be solved easily for y once the value for z is substituted in. Then the top row means $2x + 3y - z = -1$. Although it has three variables, once we know y and z this equation can be solved for x. So when a matrix has 0s in these positions, it is like a system in triangular form. Of course, the other numbers are going to change as we go through the solving process. That's because we have to follow the two rules listed above. We can't change one number in a row without changing all the others. So when we get the actual matrix in triangular form, the numbers inside will be different.

Let's go through the solving process. First, we need to change -1 in the middle row (on the far left to 0. We can do that by multiplying all of the terms in row 2 by 2. That will change the -1 to a -2.

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & -1 \\ -1 & 5 & 3 & -10 \\ 3 & -1 & -6 & 5 \end{array}\right] \quad \textbf{Multiply middle row by 2.} \longrightarrow \left[\begin{array}{ccc|c} 2 & 3 & -1 & -1 \\ \mathbf{-2} & \mathbf{10} & \mathbf{6} & \mathbf{-20} \\ 3 & -1 & -6 & 5 \end{array}\right]$$

This creates an equivalent matrix, which just means that we haven't changed the solutions. But notice that now the numbers on the far left in the top and middle rows are opposites. That means we can make the -2 into a 0 by adding the top row to the middle and making the result the new middle row.

$$\left[\begin{array}{ccc|c} \mathbf{2} & \mathbf{3} & \mathbf{-1} & \mathbf{-1} \\ \mathbf{-2} & \mathbf{10} & \mathbf{6} & \mathbf{-20} \\ 3 & -1 & -6 & 5 \end{array}\right] \quad \textbf{Add top row to middle row.} \longrightarrow \left[\begin{array}{ccc|c} 2 & 3 & -1 & -1 \\ \mathbf{0} & \mathbf{13} & \mathbf{5} & \mathbf{-21} \\ 3 & -1 & -6 & 5 \end{array}\right]$$

Next, we need to turn the 3 in the bottom row into a 0. To do that, we can make the 2 on top and the 3 on bottom opposites by multiplying the top row by 3 and the bottom row by -2. Both of these steps can be written on a scratch sheet.

top row: 3(2) 3(3) 3(-1) 3(-1) $\longrightarrow$ **6 9 -3 -3**

bottom row: -2(3) -2(-1) -2(-6) -2(5) $\longrightarrow$ **+ -6 2 12 -10**

0 11 9 -13

After getting the new numbers, we add the top row to the bottom row. That then becomes the new bottom row of the matrix.

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & -1 \\ 0 & 13 & 5 & -21 \\ \mathbf{0} & \mathbf{11} & \mathbf{9} & \mathbf{-13} \end{array}\right]$$

After you've had some practice working with matrices, you may be able to do this entire step in your head. For now, using a scratch sheet is probably safer.

We need just one more step to get the matrix in triangular form. The 11 needs to be changed to a 0. What we can do is multiply the middle row by 11 and the bottom row by -13. Here's this step on a scratch sheet.

middle row: 11(0) 11(13) 11(5) 11(-21) ➔ **0 143 55 -231**

bottom row: -13(0) -13(11) -13(9) -13(-13) ➔ **+ 0 -143 -117 169**

0 0 -62 -62

Notice that the process of making 13 and 11 opposites is a lot like finding a common denominator. We changed 13 and 11 to the product of those two numbers. The only difference is that we multiply one of the numbers by a negative, to get opposites. After replacing the old bottom row with the new sum, we end up with this.

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & -1 \\ 0 & 13 & 5 & -21 \\ \mathbf{0} & \mathbf{0} & \mathbf{-62} & \mathbf{-62} \end{array}\right]$$

Finally, the matrix is in triangular form. The bottom row has two zeros on the left, which means that the coefficients of both x and y are 0. The bottom row actually stands for the equation $-62z=-62$. In the middle row, the coefficient for x is 0. This row stands for the equation $13y+5z=-21$. The top row stands for the equation $2x+3y-z=-1$. Now all we have to do is solve for z, then y, then x.

$$-62z=-62 \qquad z=1$$

$$13y+5z=-21 \qquad 13y+5(1)=-21 \qquad 13y=-26 \qquad y=-2$$

$$2x+3y-z=-1 \qquad 2x+3(-2)-1=-1 \qquad 2x=6 \qquad x=3$$

The solutions are $x=3$, $y=-2$, and $z=1$. That's how matrices are used to solve a system.

Using a matrix is faster, because it eliminates all the extra writing and makes it easier to do certain steps in your head (once you've had some practice). And unlike the determinants shortcut, matrices are easier to use for linear systems of more than three variables. The method still works in basically the same way. But probably the biggest advantage of solving a system with matrices is that it's easier to program a computer to do the job for you. Now that's a real shortcut!

Practice 68

a. Solve the system of equations: $\begin{cases} 3x+2y-4z=-7 \\ 6y+3z=-6 \\ -3z=-12 \end{cases}$

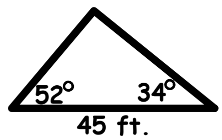
b. Solve the system of equations $\begin{cases} 14x+2y+3z=56 \\ 7x-4y+5z=41 \\ -2x+5y-2z=-26 \end{cases}$ using matrices.

c. Solve the trig equation $3\sin x - 1 = \cos 2x$ for $0 \le x \le \pi$. (All angles are in radians.)

d. If $\sin\theta = \frac{20}{29}$ and $\frac{\pi}{2} < \theta < \pi$, then $\tan\theta = ?$ Estimate your answer to two decimal places.

e. Solve the word problem below

Two birdwatchers that are 45 feet apart are watching a humming bird that is hovering directly above the path between them. The angle of elevation from the birdwatcher on the right to the bird is $34°$, and the angle of elevation from the birdwatcher on the left to the bird is $52°$. Based on this information, how far is the birdwatcher on the left from the hummingbird? Round your answer to two decimal places.



Problem Set 68

Tell whether each sentence below is True or False.

1. A matrix is a group of numbers laid out in rows and columns, shaped like a rectangle.

2. A matrix can have a different number of rows than it has columns.

If $\mathbf{t} = -3\mathbf{i} - 10\mathbf{j}$ and $\mathbf{s} = 4\mathbf{i} - 5\mathbf{j}$, calculate the value of each expression below. Estimate any irrational answers to two decimal places.

3. $4\mathbf{t} - 6\mathbf{s}$

4. $\|\mathbf{t} + \mathbf{s}\|$

For each pair of polar coordinates below, select the correct conversion into rectangular coordinates. Don't use the polar conversion function on your calculator.

5. $J(84, 240°)$

A. $(42, -42\sqrt{3})$
B. $(-42\sqrt{3}, -42)$
C. $(-42, 42\sqrt{3})$
D. $(42, 42\sqrt{3})$
E. $(-42, -42\sqrt{3})$

6. $K(130, -150°)$

A. $(-65, -65\sqrt{3})$
B. $(65\sqrt{3}, 65)$
C. $(-65\sqrt{3}, -65)$
D. $(65\sqrt{3}, -65)$
E. $(-65\sqrt{3}, 65)$

Solve each system of equations below.

7. $\begin{cases} 3x - 5y = 2 \\ 15y = 9x - 8 \end{cases}$

8. $\begin{cases} y = x^2 - 3x - 8 \\ y = -2x + 4 \end{cases}$

(a) 9. $\begin{cases} 2x - y + 3z = 19 \\ 5y + 4z = 2 \\ -2z = -6 \end{cases}$

Find the value of each determinant below.

10. $\begin{vmatrix} -7 & 5 \\ -10 & 3 \end{vmatrix}$

11. $\begin{vmatrix} -6 & -2 & 4 \\ 0 & 1 & 5 \\ 7 & -4 & -3 \end{vmatrix}$

Solve each system of equations below using determinants.

12. $\begin{cases} 4x + 2y = 2 \\ -3x - y = -4 \end{cases}$

13. $\begin{cases} 6x - 5y = -6 \\ -4x + 10y = 8 \end{cases}$

Solve each system of equations below using matrices.

14. $\begin{cases} 8x - 5y + 3z = 15 \\ 4x + y - 2z = 11 \\ -2x - 3y + 6z = -13 \end{cases}$

(b) 15. $\begin{cases} 12x + y + 4z = 2 \\ 2x - 4y + 5z = -21 \\ -3x + 2y - z = 4 \end{cases}$

Solve each trig equation below for $0 \le x < \pi$. (All angles are in radians.)

16. $\sin x \tan^2 x - 3\sin x = 0$

(c) 17. $\cos^2 x - \cos x + 1 = \cos 2x$

Select the answer for each question below.

18. All of the following functions have a period of 2 except for which one?

A. $y = 2\csc \pi x$
B. $y = 2\cos 2\pi x + 1$
C. $y = \sin \pi x + 2$
D. $y = \frac{1}{2}\cos \pi x$
E. $y = 4\sin \pi x$

19. $\dfrac{\tan\alpha + \cot\alpha}{\cot\alpha} = ?$

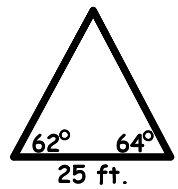
A. $\cot\alpha$
B. $\csc^2\alpha$
C. $\sec^2\alpha$
D. 1
E. $\cos^2\alpha$

20. If $(x-2)^{\frac{1}{4}} = 16$, then $x = ?$

(d) 21. If $\sin\theta = \dfrac{7}{25}$ and $\dfrac{\pi}{2} < \theta < \pi$, then $\tan\theta = ?$ Estimate your answer to two decimal places.

Solve the word problem below.

(e) 22. Two construction workers standing 25 feet apart are watching their friend on a lift directly above the path between them. The angle of elevation from the worker on the right to the worker in the lift is $64°$, and the angle of elevation from the worker on the left to the worker in the lift is $62°$. Based on this information, how far from the worker on the left is the worker in the lift? Round your answer to two decimal places.



Lesson 69—Matrix Algebra

In the last lesson, we learned how to use matrices to solve a system. Matrices aren't just used for systems, though. The concept of a matrix has many applications. It's one of the most important concepts in all of mathematics. A matrix is really just a group of numbers that go together.[3] That makes it similar to a vector. Remember, a vector can represent both the magnitude and direction of something (like a velocity or force), because it has two numbers attached to it—the two components—rather than one[4]. A matrix can contain lots more than two numbers, which means it can represent far more complicated quantities. Each of the numbers inside the matrix can stand for some different characteristic of the quantity.

Vectors can be added, subtracted, and multiplied. And the same is true for matrices. Doing operations with matrices is sometimes called "matrix algebra." So in this lesson, we'll learn how to perform some basic operations of matrix algebra.

Equal Matrices

First, though, we need to define what it means for matrices to be equal to each other. Two matrices are equal if every number inside one matrix is equal to the corresponding number inside the second matrix.

$$\begin{bmatrix} 2 & 14 & -9 \\ 5 & 7 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 14 & -9 \\ 5 & 7 & 8 \end{bmatrix}$$

Sometimes a special notation is used to represent the numbers (elements) inside a matrix. Two little numbers are placed in the bottom right corner of each element of the matrix. It looks like this: $a_{1\,2}$. This is the element a with a 1 and a 2 below. Each of these numbers is an index. The first one (the 1) stands for the row that a is in. The second one (the 2) stands for the column that a is in. So a_{12} means that a is in the first row and the second column of the matrix. More generally, we say that any element $a_{i\,j}$ is in the ith row and jth column. Here is a matrix that uses this notation for every element.[5]

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Going back to the definition of equal matrices, though, two matrices A and B are equal if each element $a_{i\,j}$ in A is equal to the corresponding element $b_{i\,j}$ in B. However, that doesn't mean that the corresponding numbers all have to be in the exact same form. Here are a couple of other examples of equal matrices where some of the numbers are written differently.

$$\begin{bmatrix} 14 & -2 & 5 \\ 9 & -6 & 17 \\ \sqrt{13} & 1 & -9 \end{bmatrix} = \begin{bmatrix} 14 & -2 & 5 \\ 9 & -6 & 17 \\ \sqrt{13} & 1 & -9 \end{bmatrix} \qquad \begin{bmatrix} 0.5 & 3 & -1 \\ 9 & \sqrt{4} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 3 & -1 \\ 9 & 2 & 0.25 \end{bmatrix}$$

Matrices with a different number of rows and or columns can never be equal. For instance, even though the top two rows of the matrices below have exactly the same numbers, the matrices still aren't equal. That's because the matrix on the right has a third row.

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[3] Technically, it's a rectangular "array" of numbers, which just means that the numbers are arranged in the shape of a rectangle.
[4] Vectors can also be 3-D, in which case they have 3 components.
[5] Notice that there's no vertical line separating any of the columns. That's because this particular matrix isn't being used to solve a system.

$$\begin{bmatrix} -6 & 3 & -1 \\ 9 & \sqrt{4} & 0.25 \end{bmatrix} \neq \begin{bmatrix} -6 & 3 & -1 \\ 9 & \sqrt{4} & 0.25 \\ 0.8 & 2 & -7 \end{bmatrix}$$

Adding and Subtracting Matrices

The process of adding two matrices is actually pretty easy. All we have to do is add all of the corresponding numbers. Here's an example of a matrix *A* added to another matrix *B*.

$$\begin{array}{ccc} A & & B \\ \begin{bmatrix} 5 & -8 & 1 \\ 2 & 4 & 10 \\ -1 & 19 & -6 \end{bmatrix} & + & \begin{bmatrix} 15 & -3 & 7 \\ -1 & -6 & -21 \\ 8 & -7 & -2 \end{bmatrix} \end{array}$$

We just add 5 and 15 to get 20. And that sum goes in the first row and first column of the sum. Then we add -8 and -3 to get -11, which goes in the first row and second column of the answer, and so on until we've added every pair of numbers. The answers go together to create a new 3 by 3 matrix, which is the sum of the two original matrices.

$$A+B = \begin{bmatrix} 5+15 & -8+-3 & 1+7 \\ 2+-1 & 4+-6 & 10+-21 \\ -1+8 & 19+-7 & -6+-2 \end{bmatrix} = \begin{bmatrix} 20 & -11 & 8 \\ 1 & -2 & -11 \\ 7 & 12 & -8 \end{bmatrix}$$

Generally, for any matrix *A* and *B*, the sum $A+B$ is equal to the sum of the corresponding elements of $a_{i\,j}$ in *A* and $b_{i\,j}$ in *B*.

What about subtracting matrices? As you might expect, we just subtract the corresponding numbers. Here's a subtraction example.

$$\begin{array}{ccc} A & & B \\ \begin{bmatrix} 9 & 2 & -3 \\ 18 & 1 & -7 \\ 3 & 14 & -5 \end{bmatrix} & - & \begin{bmatrix} 2 & 5 & 4 \\ -5 & 25 & -6 \\ 4 & 11 & 9 \end{bmatrix} \end{array}$$

We just take $9-2$, which equals 7. Then we do $2-5$, which is -3. The process continues for all nine pairs of numbers.

$$A-B = \begin{bmatrix} 9-2 & 2-5 & -3-4 \\ 18--5 & 1-25 & -7--6 \\ 3-4 & 14-11 & -5-9 \end{bmatrix} = \begin{bmatrix} 7 & -3 & -7 \\ 23 & -24 & -1 \\ -1 & 3 & -14 \end{bmatrix}$$

There's one important point about adding and subtracting matrices. To do the calculation, the matrices must have the same number of rows and columns. It always has to be $m\times n+m\times n$ or $m\times n-m\times n$.

It's also kind of interesting that matrices follow the commutative property of addition and the associative property of addition. Remember, the commutative property says that for any two numbers *a* and *b* that $a+b=b+a$. And the associative property of addition says that $(a+b)+c=a+(b+c)$. Both of these rules also hold for any two matrices *A* and *B*, as long as the number of rows and columns of the matrices are the same, of course.

Commutative Property of Addition for Matrices

$$A + B = B + A$$

Associative Property of Addition for Matrices

$$(A + B) + C = A + (B + C)$$

Multiplication with Matrices

Sometimes a matrix will need to be multiplied by a real number (a scalar). To do that, we just multiply each of the numbers inside the matrix by the scalar. The products become the numbers in the new matrix. As an example, if we multiply matrix A by 2, the product matrix P is calculated like this.

$$\begin{array}{cccc} & A & & P \\ 2 & \begin{bmatrix} 5 & -8 & 1 \\ 2 & 4 & 10 \\ -1 & 19 & -6 \end{bmatrix} = & \begin{bmatrix} 2(5) & 2(-8) & 2(1) \\ 2(2) & 2(4) & 2(10) \\ 2(-1) & 2(19) & 2(-6) \end{bmatrix} = & \begin{bmatrix} 10 & -16 & 2 \\ 4 & 8 & 20 \\ -2 & 38 & -12 \end{bmatrix} \end{array}$$

Notice that $3(a_{11}) = p_{11}$, $3(a_{12}) = p_{12}$, $3(a_{13}) = p_{13}$, and so on.

It's also possible to multiply two matrices. The most obvious way to do it is to just multiply each pair of corresponding numbers. But that turns out to be wrong. Multiplying matrices is actually more complicated than that. To show you how multiplication works, let's say we need to multiply the two matrices below.

$$\begin{array}{cc} A & B \\ \begin{bmatrix} 3 & 10 & -7 \\ 4 & -1 & 9 \end{bmatrix} & \begin{bmatrix} 5 & 8 & -2 & 1 \\ 6 & 12 & 4 & 2 \\ 3 & 9 & -6 & -3 \end{bmatrix} \end{array}$$

First of all, notice that A is a 2 by 3 matrix and B is a 3 by 4 matrix. We can still multiply these two. What matters when multiplying matrices is that the number of columns in the first matrix has to equal the number of rows in the second matrix. There are 3 columns in A and 3 rows in B, so that works. Now what we do is multiply each of the numbers in the first row of A by each of the numbers in the first column of B and add the results.

Multiply row 1 of A by column 1 of B

$$3(5) + 10(6) + -7(3) = 54$$

This answer, 54, becomes the first number (in the upper left) in the matrix of the product matrix AB. To get the second number in the product, we multiply the first row of A by the second column of B.

Multiply row 1 of A by column 2 of B

$$3(8) + 10(12) + -7(9) = 81$$

This answer, 81, is the second number in the top row of the product matrix. To get the third and fourth numbers in the top row of the product, we multiply the first row of A by the third column of B and then by the fourth column of B.

Multiply row 1 of A by columns 3 and 4 of B

$$3(-2)+10(4)+-7(-6)=76$$

$$3(1)+10(2)+-7(-3)=44$$

So the first row of the product matrix is 54, 81, 76, 44. To calculate the second row of the product matrix, we go through the same process, only multiplying by row 2 of *A*.

Multiply row 2 of A by all columns of B

by column 1: $4(5)+-1(6)+9(3)=41$

by column 2: $4(8)+-1(12)+9(9)=101$

by column 3: $4(-2)+-1(4)+9(-6)=-66$

by column 4: $4(1)+-1(2)+9(-3)=-25$

The second row of the product matrix is 41, 101, -66, -25. Putting both rows together gives us the product matrix below.

$$AB=\begin{bmatrix} 54 & 81 & 76 & 44 \\ 41 & 101 & -66 & -25 \end{bmatrix}$$

Notice that we multiplied by 2 by 3 matrix by a 3 by 4 matrix and came out with a 2 by 4 product matrix. The product has the same number of rows as the first matrix and the same number of columns as the second matrix. Generally, to multiply two matrices, the inside numbers have to be the same. In other words, you can multiply an $m\times n$ matrix by an $n\times p$ (the *n*'s are the same). Then the answer will be an $m\times p$ matrix. Those are the outside numbers. Here's how to find each element of a product matrix, written out in words.

$$AB=\begin{bmatrix} \text{Row 1 of } A & \text{Row 1 of } A & \text{Row 1 of } A & \text{Row 1 of } A \\ \text{times column} & \text{times column} & \text{times column} & \text{times column} \\ \text{1 of } B & \text{2 of } B & \text{3 of } B & \text{4 of } B \\ & & & \\ \text{Row 2 of } A & \text{Row 2 of } A & \text{Row 2 of } A & \text{Row 2 of } A \\ \text{times column} & \text{times column} & \text{times column} & \text{times column} \\ \text{1 of B} & \text{2 of } B & \text{3 of } B & \text{4 of } B \end{bmatrix}$$

For matrices with different numbers of rows and columns, the pattern works basically the same way.

As we said, the number of columns in the first matrix must equal the number of rows of the second matrix when multiplying. (In other words, the inside numbers have to be the same). Because of this requirement, multiplication with matrices is not commutative.

Matrix multiplication is not commutative.

$$A \times B \neq B \times A$$

Think about it. In our example, we multiplied a 2×3 matrix by a 3×4 matrix. That worked, because the inside numbers were the same (3 and 3). But what if we had tried to multiply $B\times A$? That's a 3×4 matrix by a 2×3 matrix, which wouldn't have worked, because the inside numbers are different (4 and 2). If both matrices have the same number of rows as columns (matrices of this type are called square matrices, by the way.), then both $A\times B$ and $B\times A$ will have answers. But the answers won't necessarily be the same. For instance, matrices *A* and *B* below are both 2×2, but *AB* doesn't equal *BA*.

$$AB = \begin{bmatrix} 2 & 9 \\ -7 & 5 \end{bmatrix} \begin{bmatrix} 4 & 11 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 53 & -5 \\ -3 & -92 \end{bmatrix}$$

$$BA = \begin{bmatrix} 4 & 11 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} 2 & 9 \\ -7 & 5 \end{bmatrix} = \begin{bmatrix} -69 & 91 \\ 31 & 30 \end{bmatrix}$$

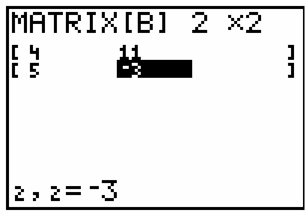
$$AB \neq BA$$
$$\begin{bmatrix} 53 & -5 \\ -3 & -92 \end{bmatrix} \neq \begin{bmatrix} -69 & 91 \\ 31 & 30 \end{bmatrix}$$

Matrices on a Calculator

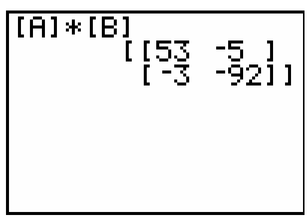
A calculator can make adding, subtracting, and multiplying matrices a lot easier. To enter a matrix on the calculator, we press $\boxed{\text{2nd}}$, $\boxed{x^{-1}}$, which is the "matrix" key. The calculator calls the first matrix that we enter, matrix *A*. Now we press the right arrow key twice, to get to the "edit" menu. Next, we press $\boxed{\text{ENTER}}$. Now we have to tell the calculator what the dimensions of the matrix are. To enter a 2 by 2 matrix, we press 2, then press the right arrow and press 2 again, then $\boxed{\text{ENTER}}$.

The next step is to enter the elements of the matrix in their proper places. To enter $\begin{bmatrix} 2 & 9 \\ -7 & 5 \end{bmatrix}$, we press 2, then $\boxed{\text{ENTER}}$. The calculator automatically moves us to the next position. It also tells us which position we're in at the bottom of the screen. After 2, we press 9 and $\boxed{\text{ENTER}}$, then -7, $\boxed{\text{ENTER}}$ and 5, $\boxed{\text{ENTER}}$.

Now to enter a second matrix $\begin{bmatrix} 4 & 11 \\ 5 & -3 \end{bmatrix}$, we press $\boxed{\text{2nd}}$, $\boxed{\text{MODE}}$, which means "quit." That tells the calculator we're starting over. Next, we press $\boxed{\text{2nd}}$, $\boxed{x^{-1}}$, which takes us back to the main matrix screen. We press the right arrow twice to get to "edit." Then we press the down arrow to get to matrix *B* and press $\boxed{\text{ENTER}}$. At this point, the calculator asks us to tell the dimensions of the matrix, which is 2 by 2 again. So we press 2, right arrow and 2 again, followed by $\boxed{\text{ENTER}}$. Now we put in the four elements of this second matrix, pressing $\boxed{\text{ENTER}}$ after each one.



To multiply matrix *A* and matrix *B*, we press $\boxed{\text{2nd}}$, $\boxed{\text{MODE}}$ to quit and $\boxed{\text{2nd}}$, $\boxed{x^{-1}}$ to go back to the matrix screen. Now, with the cursor on matrix *A*, we press $\boxed{\text{ENTER}}$. Then we press the multiplication key. Next, we press $\boxed{\text{2nd}}$, $\boxed{x^{-1}}$ to go back to the matrix screen. To select matrix *B*, we press the down arrow and $\boxed{\text{ENTER}}$. That causes the calculator to show [A]*[B] . The last step is to press $\boxed{\text{ENTER}}$ again, to actually perform the operation. The calculator shows the product matrix.



Adding and subtracting matrices work the same way, only instead of pressing the multiplication key, you press the addition or subtraction key.

Practice 69

a. Solve the system of equations $\begin{cases} 6x+4y-2z=14 \\ -4x-2y+5z=-4 \\ 5x+y-6z=10 \end{cases}$ using determinants.

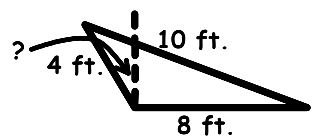
b. If $A=\begin{bmatrix} -6 & 9 & 5 \\ 8 & 6 & -17 \\ -5 & 13 & 8 \end{bmatrix}$ and $C=\begin{bmatrix} 4 & 9 & -2 \\ 0 & -5 & 6 \end{bmatrix}$, calculate the value of CA.

c. Use your knowledge of trig identities (not a calculator) to find each trig value below.
If $\cos\alpha=-\frac{5}{7}$, find $\sin(\alpha-\frac{\pi}{2})$.

d. If $\cos(18+6x)=\sin(3x)$, then $x=?$

A. 5° B. 8° C. 10°
D. 18° E. 28°

e. After a recent tornado, a light post was leaning to the side. As a temporary solution until a new post came in, workers drove a stake into the ground 8 feet from the post and attached a 10-foot cable from the stake to the post 4 feet from the bottom of the post. Based on this information, how far from the vertical was the post leaning? Round your answer to two decimal places.



Problem Set 69

Tell whether each sentence below is True or False.

1. It's possible to add or subtract matrices that have different numbers of rows and columns.

2. It's possible to multiply matrices that have different numbers of rows and columns.

For each pair of parametric equations below, eliminate the parameter and select the direct relationship between x and y.

3. $x=6-\frac{3}{5}t$, $y=3t$

A. $y=6-\frac{9}{5}x$ B. $y=18-\frac{9}{5}x$ C. $y=6-\frac{1}{5}x$
D. $y=18x-\frac{9}{5}x^2$ E. $y=30-5x$

4. $x=5-t$, $y=2t^2-3t$

A. $y=-2x^2+3x+5$ B. $y=-2x^3+13x^2-15x$ C. $y=2x^2-3x$
D. $y=2x^2-17x+35$ E. $y=5-x$

Solve each system of equations below.

5. $\begin{cases} x^2 + y^2 = 53 \\ 7y = 2x \end{cases}$

6. $\begin{cases} 3x + 2y + 5z = 0 \\ 4y - 6z = 52 \\ -3z = 12 \end{cases}$

Solve each system of equations below using determinants.

7. $\begin{cases} 3x - 5y = 47 \\ 7x + 4y = 0 \end{cases}$

8. $\begin{cases} -12x + 9y = 8 \\ 6x - 15y = -11 \end{cases}$

(a) 9. $\begin{cases} 10x - 4y + z = 30 \\ -5x + y + 3z = -2 \\ 2x - 3y - 5z = -17 \end{cases}$

If $A = \begin{bmatrix} -2 & 6 & 8 \\ 3 & 9 & -15 \\ -4 & 12 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -5 & 14 \\ -7 & 0 & 6 \\ -3 & -11 & 9 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 6 & -1 \\ 0 & -2 & 4 \end{bmatrix}$, calculate the value of each expression below.

10. $A + B$

11. $B - A$

12. $3B$

(b) 13. CA

Use your knowledge of trig identities (not a calculator) to find each trig value below.

(c) 14. If $\cos\theta = -\dfrac{3}{4}$, find $\sin(\theta - \dfrac{\pi}{2})$

15. If $\sin 2\alpha = \dfrac{2}{3}$, find $\dfrac{1}{\sin\alpha\cos\alpha}$

Select the answer for each question below.

16. If $f(x) = x^2 + 3$ for all $x \geq 0$, then the graph of $f^{-1}(x)$ intersects the x-axis at ?

A. $f^{-1}(x)$ is undefined
B. exactly three points
C. exactly two points
D. exactly one point
E. zero points

17. Which of the following is the equation of a line with x-intercept $(-4, 0)$ and y-intercept $(0, -6)$?

A. $6x + 4y = 24$
B. $4x + 6y = -24$
C. $6x - 4y = -24$
D. $-6x + 4y = 24$
E. $6x + 4y = -24$

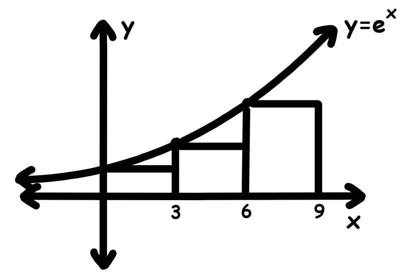
(d) 18. If $\cos(65 + 3x) = \sin(2x)$, then $x = ?$

A. 0°
B. 5°
C. 10°
D. 25°
E. 30°

Answer the problems below.

19. What is the value of n if the determinant of X is given as: $\det X = \begin{vmatrix} 9 & -3 \\ n & 5 \end{vmatrix} = 0$?

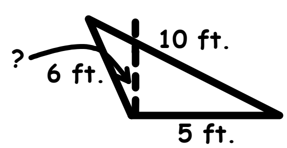
20. A portion of the graph of $y = e^x$ is shown in the figure to the right. What is the sum of the areas of the three inscribed rectangles? Estimate your answer to two decimal places.



21. If $f(x) = 5^x$ for all real values of x and $f^{-1}(k) = -3$, then $k = ?$

Solve the word problem below.

(e) 22. A hurricane partially uprooted a tree in the Jones' front yard so that it was slanted from its normal upright position. To prevent it from falling on the house until Mr. Jones could cut it down, he drove a stake into the ground 5 feet from the tree and then ran a 10-foot rope from the stake to a place in the tree 6 feet from the base of the tree. Based on this information, how far from the vertical was the tree leaning?



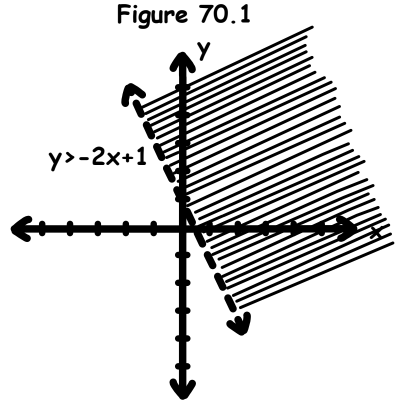
Lesson 70—Systems of Inequalities

We've learned a lot in this chapter about systems of equations. But there are also systems of inequalities. Some inequalities have just one variable. An example is $3x-5\le 4$. To solve these, all you have to remember is to flip the inequality symbol when multiplying or dividing both sides by a negative number. Other than that, the solving process works just like one-variable equations. The solutions to $-3x-5\le 4$ are $x\ge -3$, which means x can equal -3 or anything greater than -3.

Two-Variable Inequalities

Inequalities can also have two variables. The inequality $y>-2x+1$ is a two-variable example. It tells us the relationship between the variables x and y. Only instead of the solutions being the pairs of numbers that make both sides equal, they're the pairs of numbers that make the left side greater than the right side. The simplest way to view the solutions of a two-variable inequality is to graph it. The first step is to graph the *equation* $y=-2x+1$. The equation is linear, so the graph is a straight line. We'll use a dashed line instead of a solid line. The line has to be dashed, because all of the points on the line represent solution pairs that make y and $-2x+1$ equal to each other. The graph of $y>-2x+1$ includes only points where y is greater than $-2x+1$. So the points on the line aren't included in the graph, but we need those points to find the real solutions to the inequality. The points that make y greater than $-2x+1$ are all above the line. So to show the solutions to the inequality, we shade that entire region.

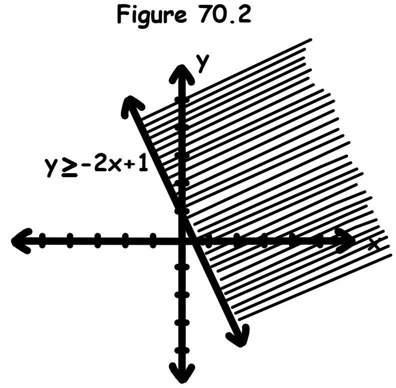
Figure 70.1



You can check to make sure that these points work. Just choose any one of them and substitute the x and y values into the inequality. You'll see that it makes the left side greater than the right side.

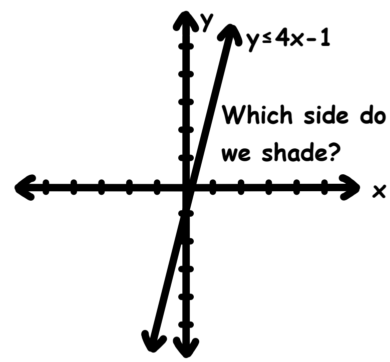
If the inequality has required y to be greater than *or equal to* $-2x+1$, then the solutions would have included all the points above the line as well as the points on the line itself. To graph the inequality $y\ge -2x+1$, we would need to make the line solid to show that those points are included.

Figure 70.2



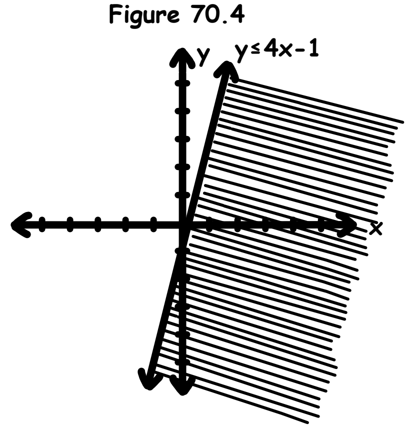
Sometimes it can be a little hard to tell which side of the line the pairs of solutions to an inequality are on. For instance, the inequality $y \le 4x-1$ includes all the pairs on the line $y = 4x-1$ and all the pairs below that line. But if you look at the graph, the line is really steep. Which side is above and which side is below?

Figure 70.3



The easiest way to tell is just pick a point on one side or the other and substitute it into the inequality. If it works, then you know automatically that all of the other points on that side of the line will work also. So you can shade that side. If it doesn't work, then you know that none of the points on that side will work, and you can shade the other side. The easiest point to choose is usually the origin, since it makes the arithmetic really simple. Substituting $(0,0)$ into $y \le 4x-1$ gives $0 \le 4(0)-1$ or $0 \le -1$, which is false. That means we have to shade the other side of the line.

Figure 70.4



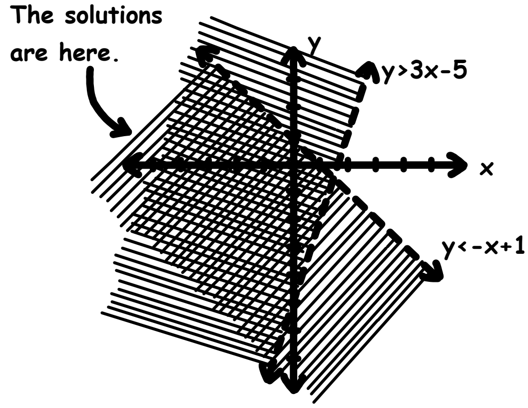
Intersecting Regions

As you might expect, systems of inequalities are just groups of inequalities that go together. The solutions are all the pairs of numbers that solve all the inequalities in the system. Here's a simple example.

$$\begin{cases} y > 3x-5 & \textbf{Inequality 1} \\ y < -x+1 & \textbf{Inequality 2} \end{cases}$$

The solutions here are the pairs of numbers that solve both $y > 3x-5$ and $y < -x+1$. The easiest way to see the solutions of a system of inequalities is to graph it. What we do is graph both inequalities on the same coordinate plane. Then the solutions are where the shaded regions overlap. That makes sense, because the overlap is all the pairs that solve both inequalities.

Figure 70.5

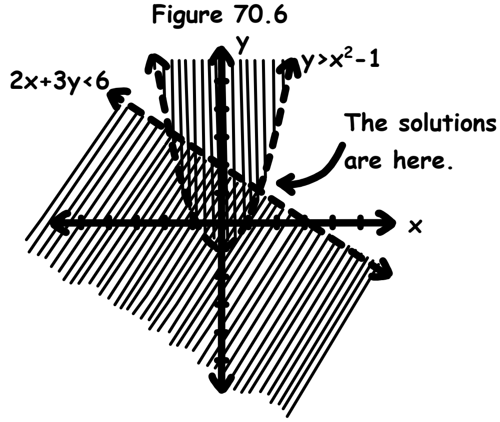


Some systems have nonlinear inequalities. Here's an example like that.

$$\begin{cases} y > x^2 - 1 & \textbf{Inequality 1} \\ 2x + 3y < 6 & \textbf{Inequality 2} \end{cases}$$

See, Inequality 1 is second-degree. Its graph is a parabola. To find the solutions, we still just graph both inequalities on the same coordinate plane and look for the region of intersection. First, we graph $y = x^2 - 1$, which is a parabola opening upward. We need to use a dashed line, though, since the inequality is greater than, not greater than or equal to. Next, we graph $2x + 3y = 6$, which is the same as $y = -\frac{2}{3}x + 2$. This line should also be dashed. The shading for $y > x^2 - 1$ should go above the curve, and the shading for $y < -\frac{2}{3}x + 2$ should go below the line.

Figure 70.6



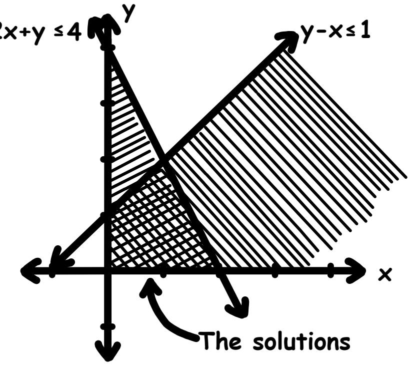
The region of intersection represents all the solutions to the system. Those are the pairs that will solve both inequalities.

It's possible for a system to contain more than two inequalities. Look at this one.

$$\begin{cases} 2x + y \le 4 & \textbf{Inequality 1} \\ y - x \le 1 & \textbf{Inequality 2} \\ x \ge 0 & \textbf{Inequality 3} \\ y \ge 0 & \textbf{Inequality 4} \end{cases}$$

Inequalities 3 and 4 are basically limiting the variables *x* and *y* to only positive values. That means the graph of this system will fall completely in the first quadrant of the coordinate plane. Here's what the graph looks like.

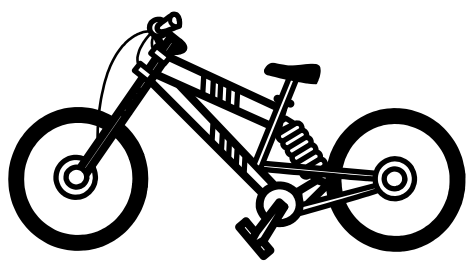
Figure 70.7



Notice both lines are solid, since the inequalities are less than or equal to. And the shading has to go below the lines in both cases. Because of Inequalities 3 and 4, we only shade up to the *x* and *y*-axis, no further. As always, the intersecting region represents the solutions to the system.

Linear Programming

One important real-world use for systems of inequalities is in the field of business. That's because business owners are always interested in maximizing their profits. There's an area of math called linear programming which helps people maximize or minimize a certain quantity. Linear programming uses systems of inequalities. Let's go through an example.



> Pedal Master, Inc. makes two kinds of bikes: mountain bikes and road bikes. A mountain bike takes 2 hours to manufacture and 2 hours to paint, while a road bike takes 4 hours to manufacture and 2 hours to paint. The company's factory is big enough to do up to 40 work-hours per day of manufacturing and up to 32 work-hours per day of painting. If Pedal Master earns \$250 on each road bike and \$200 on each mountain bike, how many of each should be produced per day to maximize profit?

We'll let *P* be Pedal Master's profits, and we'll let *x* be the number of mountain bikes it makes each day, and *y* the number of road bikes it makes each day. Putting those variables together, we can write an equation to represent the company's profits per day.

profits **mountain bikes** **road bikes**

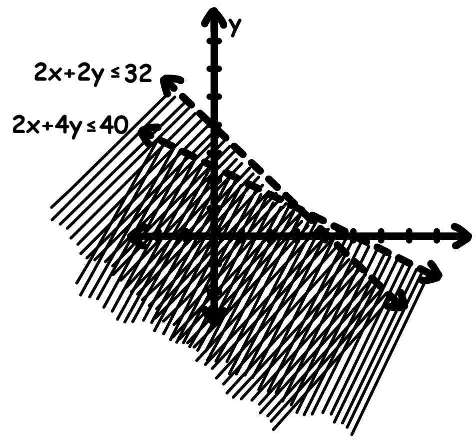
$$P = 200x + 250y$$

If there were no limits on the number of bikes that Pedal Master could produce and sell, then their profits (*P*) could go to infinity. But, remember, the company's factory can only produce so much in a day. We can write two-variable inequalities to represent the factory's production limits. A mountain bike takes 2 hours to manufacture and 2 hours to paint, and a road bike takes 4 hours to manufacture and 2 hours to paint. That means we can represent the amount of time spent manufacturing mountain bikes each day as $2x$ and the amount of time spent manufacturing road bikes each day as $4y$. Since the factory's manufacturing time can't be more than 40 hours of work per day, we know that $2x+4y \le 40$. There are also limits on the amount of painting that can be done. A mountain bike takes 2 hours to paint and so does a road bike. In a day, then, the amount of time spent painting mountain bikes can be represented as $2x$ and the amount of time spent painting road bikes as $2y$. Since the painting time can't be more than 32 hours of work per day, that gives us $2x+2y \le 32$. Finally, since it doesn't make any sense to produce a negative number of products, the variables x and y should be prevented from taking on negative values. That gives us the inequalities $x \ge 0$ and $y \ge 0$. All four of these inequalities represent the same problem, so we can put them together as a system.

$$\begin{cases} 2x+4y \le 40 \\ 2x+2y \le 32 \\ x \ge 0 \\ y \ge 0 \end{cases}$$

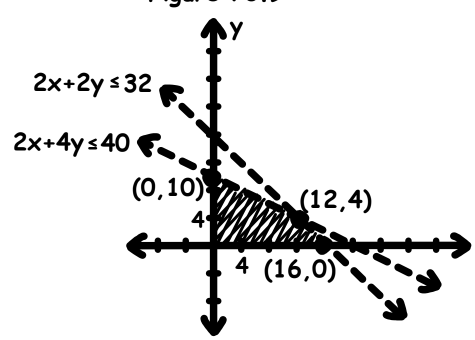
The solutions to this system can be shown by graphing all four inequalities on the same coordinate plane and shading properly.

Figure 70.8



Notice that the graph is only in the first quadrant, because of $x \ge 0$ and $y \ge 0$. The solutions to the inequalities are in the region where the shading overlaps. It's easier to see this region if we eliminate all of the other shading in the graph.

Figure 70.9



Remember, our goal is to maximize P in the equation $P = 200x + 250y$. So what we need to do next is figure out which of the pairs of solutions for x and y give us the largest value for P in $P = 200x + 250y$. We have to use pairs of solutions to the system of inequalities, because other pairs don't fall within the company's production limits. As it turns out, the maximum values for P must fall on one of the corners of the solution region. This fact has been proven by mathematicians. So we can just plug in the pairs of solutions at the three corners of the solutions region into the equation $P = 200x + 250y$ and compare them. Those pairs are $(0,10)$, $(12,4)$, and $(16,0)$.

(0,10)	**(12,4)**	**(16,0)**
$P = 200(0) + 250(10)$	$P = 200(12) + 250(4)$	$P = 200(16) + 250(0)$
$P = 2{,}500$	$P = 3{,}400$	$P = 3{,}200$

The pair that gives the highest value for P is $(12,4)$, which means that profit is maximized by producing 12 mountain bikes and 4 road bikes per day.

We just solved a "linear programming" problem. Linear programming can be used to maximize and minimize certain quantities. It's particularly important in business, where companies are always wanting to maximize profits and minimize costs. But linear programming has many other uses as well. Linear programming is just one example of how systems of inequalities are used in the real world.

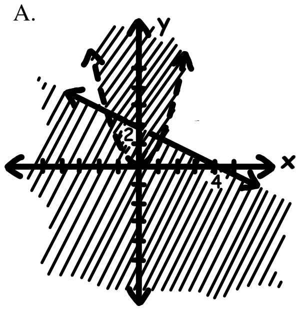
Practice 70

a. Solve the system of equations $\begin{cases} 5x - 2y + 6z = 35 \\ -10x + 6y - 2z = -58 \\ 2x - 4y + 5z = 32 \end{cases}$ using determinants.

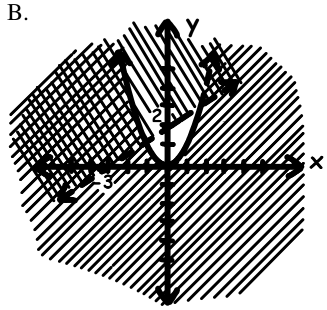
b. If $B = \begin{bmatrix} 8 & 4 & -17 \\ -7 & 10 & 6 \\ 12 & -9 & 15 \end{bmatrix}$ and $C = \begin{bmatrix} -9 & 14 & 5 \\ 3 & -7 & 12 \end{bmatrix}$, calculate the value of CB.

c. Select the graph for the system of inequalities: $\begin{cases} y \le x^2 \\ x + 2y \le 4 \end{cases}$

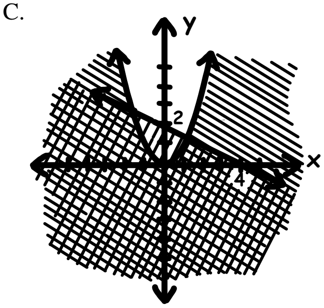
A.



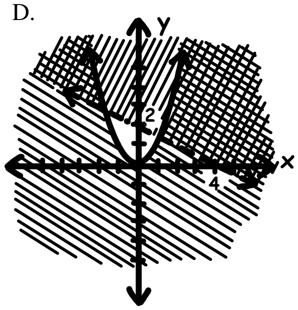
B.



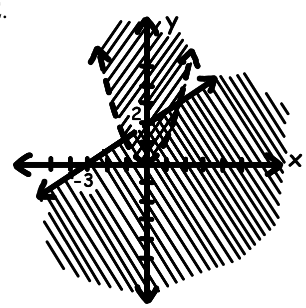
C.



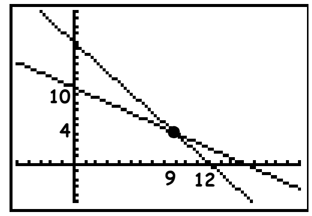
D.



E.



d. Joe bought 96 ounces of fresh peaches and 60 ounces of canned peaches to make peach cobbler for a bake sale. He has a recipe for regular peach cobbler which calls for 6 ounces of fresh peaches mixed with 6 ounces of canned peaches and a recipe for extra special peach cobbler which calls for 8 ounces of fresh peaches mixed with 4 ounces of canned peaches. If Joe sells his regular peach cobbler for \$2.00 a cobbler and his extra special peach cobbler for \$2.15 a cobbler then how many of each should he make to maximize his profit? (In the figure on the right, x stands for the amount of extra special and y stand for the amount of regular.)



e. Which of the following is an asymptote of $f(x) = \tan 4x$?

A. $x = \dfrac{\pi}{8}$ B. $x = \dfrac{\pi}{6}$ C. $x = \dfrac{\pi}{4}$

D. $x = \dfrac{\pi}{2}$ E. $x = \pi$

f. A rectangular prism has three different side lengths, x, y, and z. The perimeter of the face with sides x and y is 28 centimeters, and the perimeter of the face with sides x and z is 32. If the perimeter of the face with sides y and z is 36, what is the volume of the prism?

Problem Set 70

Tell whether each sentence below is True or False.

1. The easiest way to view the solutions to a system of inequalities is to graph it.

2. Linear programming is an area of math that helps people maximize or minimize a certain quantity.

Solve each system of equations below.

3. $\begin{cases} 3x - 4y = 8 \\ 8y = 6x - 16 \end{cases}$

4. $\begin{cases} xy = 6 \\ x - y + 1 = 0 \end{cases}$

Solve each system of equations below using determinants.

5. $\begin{cases} -6x + 2y = -30 \\ 8x - 5y = 47 \end{cases}$

6. $\begin{cases} 3x + 14y = -4 \\ -9x - 7y = 7 \end{cases}$

(a) 7. $\begin{cases} 4x + 6y - 3z = -13 \\ -12x - y + 5z = -16 \\ 6x + 3y + 4z = 7 \end{cases}$

If $A = \begin{bmatrix} 5 & 2 & -16 \\ -4 & 7 & 3 \\ 10 & -9 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -9 & 15 \\ 8 & 7 & -10 \\ -11 & 2 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 3 & 5 \\ -4 & 0 & -7 \end{bmatrix}$, calculate the value of each expression below.

8. $A + B$

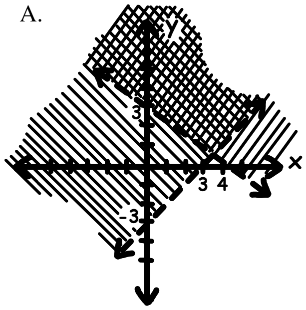
9. $A - B$

10. $-2C$ **(b) 11.** CB

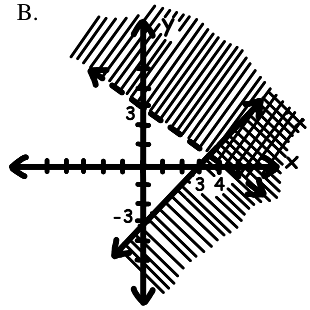
Select the graph for each system of inequalities below.

12. $\begin{cases} 3x+4y>12 \\ -x+y\le -3 \end{cases}$

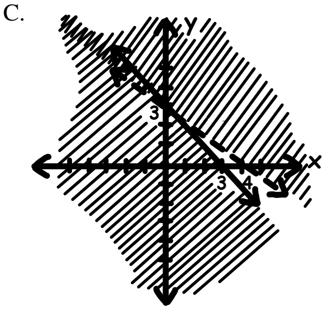
A.



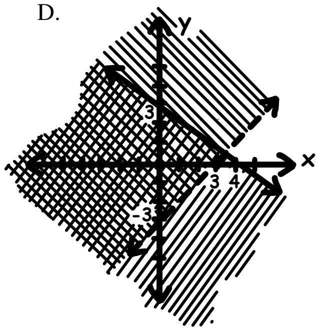
B.



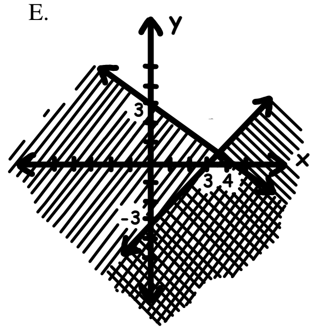
C.



D.

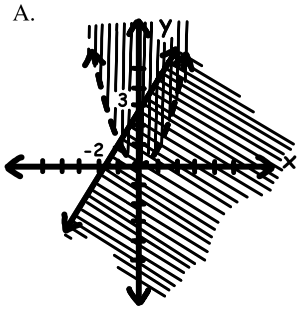


E.

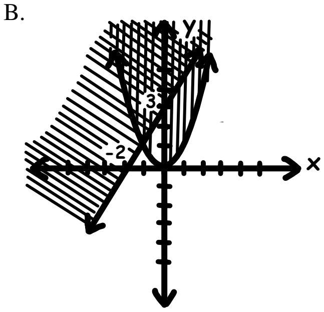


(c) 13. $\begin{cases} y\le x^2 \\ 3x-2y\ge -6 \end{cases}$

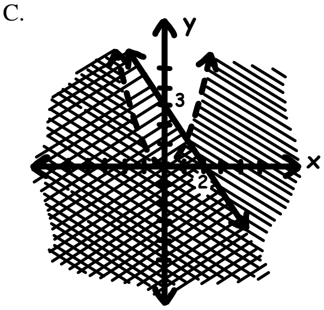
A.



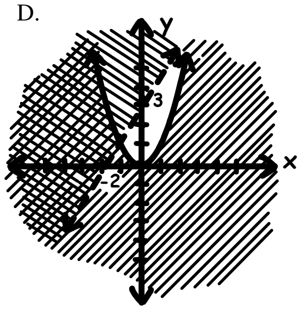
B.



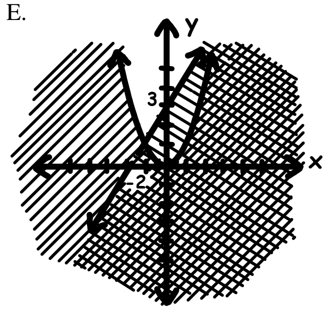
C.



D.

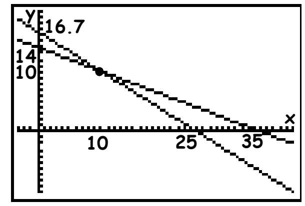


E.

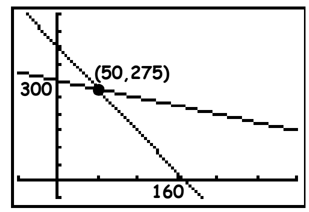


Solve each problem below.

14. A store sells prints and photographs. A photograph costs \$20 to purchase and it takes 2 hours to frame each photograph. A print costs \$30 to purchase and it takes 5 hours to frame each print. The store has at most \$500 to spend and at most 70 hours to frame. If the store makes \$25 profit on each photograph and \$40 profit on each print, find the number of each that it should purchase to maximize profits. (In the figure to the right, x stands for number of photographs and y stand for number of prints.)

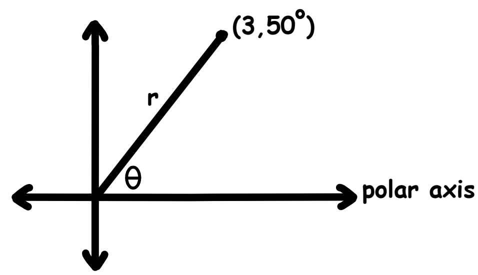


(d) 15. Mixed Up Nuts, a company which produces mixed nuts, receives a shipment of 1,600 ounces of walnuts and 3,600 ounces of almonds every day. The manager, Larry, decides he wants to try a new mixture, consisting of only walnuts and almonds. He makes a lower-priced version consisting of 10 ounces of walnuts and 6 ounces of almonds and a quality version consisting of 4 ounces of walnuts and 12 ounces of almonds. If Larry decides to sell the low-price version for \$2.00 and the quality version for \$3.00, then how much of each should be made daily to maximize the profit. (In the figure on the right, x stands for the amount of the lower-priced mixture and y stands for the amount of the quality mixture.)



Select the answer for each question below.

16. In the figure on the right, the polar coordinates of the point shown are $(3, 50°)$. What is the value of r?



A. $3\cos\theta$ B. $3\sin\theta$
C. 2.3 D. 3
E. 4.

(e) 17. Which of the following is an asymptote of $f(x) = \tan 3x$?

A. $x = \pi$ B. $x = \frac{3\pi}{4}$ C. $x = \frac{\pi}{3}$

D. $x = \frac{\pi}{4}$ E. $x = \frac{\pi}{6}$

18. For $x \neq -2$ and $x \neq \frac{1}{4}$, if $f(x) = 1 - 4x$ and $g(x) = 4x^2 + 7x - 2$, then $\frac{f(x)}{g(x)} = ?$

A. $-x-2$ B. $\frac{1}{x+2}$ C. $-\frac{1}{x+2}$

D. $\frac{-1}{x-2}$ E. $4x^2 + 11x - 3$

Solve each problem below.

19. What is the maximum value of the function $f(x) = \frac{1}{x}$ over the interval $\frac{1}{3} \le x \le \frac{5}{2}$?

20. If $x = \arctan(-\frac{1}{5})$ and $x + y = 290°$, then $\cos y = ?$ Estimate your answer to two decimal places.

21. If $f(x) = e^{3x}$, then $f(\ln 3) = ?$

Solve the word problem below.

(f) 22. A rectangular prism has three different side lengths, *a*, *b*, and *c*. The perimeter of the face with sides *a* and *b* is 32 centimeters, and the perimeter of the face with sides *a* and *c* is 20. If the perimeter of the face with sides *b* and *c* is 24, what is the volume of the prism?